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Transonic magnetohydrodynamic flows

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Stationary flows of an ideal plasma with translational symmetry along the (vertical) z axis are considered, and it is demonstrated how they can be described in the intrinsic (natural) coordinates (ξ, η, ϑ), where ξ is a label of flux and stream surfaces, η is the total pressure and ϑ is the angle between the horizontal magnetic (and velocity) field and the x axis. Three scalar nonlinear equilibrium equations of mixed elliptic–hyperbolic type for $\vartheta(\xi, \eta)$, $\xi(\eta, \vartheta)$ and $\eta(\vartheta, \xi)$ respectively are derived. The equilibrium equation for $\vartheta(\xi, \eta)$ is especially useful, and has considerable advantages compared with the coupled system of algebraic–differential equations that are conventionally used for studying plasma flows. In particular, for this equation the location of the regions of ellipticity and hyperbolicity can be determined a priori. Relations between the equilibrium equation for $\vartheta(\xi, \eta)$ and the nonlinear hodograph equation for $\xi(\eta, \vartheta)$ are elucidated. Symmetry properties of the intrinsic equilibrium equations are discussed in detail and their self-similar solutions are described. In particular, magnetohydrodynamic counterparts of several classical flows of an ideal fluid (the Prandtl–Meyer flows around a corner, the spiral flows and the Ringleb flows around a plate, etc.) are found. Stationary flows described in this paper can be used for studying both astrophysical and thermonuclear plasmas.

1. Introduction

The investigation of plasma equilibria is one of the most important problems of magnetohydrodynamics, and arise a variety of fields, such as thermonuclear fusion, astrophysics, solar physics and geophysics, to mention just a few (see e.g. Priest 1982; Goedbloed 1983; Freidberg 1987; Lifschitz 1989).

One can distinguish static equilibria (i.e. equilibria without mass motions), for which only the magnetic field, pressure and (possibly) density have to be determined, and steady plasma flows, which involve mass motions, so that, in addition to the magnetic field etc., the velocity field has to be determined as well.

At present, difficulties associated with describing fully three-dimensional equilibrium configurations, be they static or otherwise, are far from being resolved (Garabedian 1983; Bauer *et al.* 1984). For that reason, considering configurations with additional symmetries is imperative from the mathematical viewpoint. Fortunately, these configurations are the most interesting and important ones from the physical viewpoint as well. In thermonuclear fusion (tokamaks) and in many astrophysical situations (solar wind, mass outflows from young stellar objects, compact stellar objects and active galactic nucleis, accretion discs, etc.) axial symmetry is appropriate, in solar physics (evolution of solar arcades) translational symmetry is a dominant one, and, finally, the most general helical symmetry which embraces the other two as limiting cases, is of interest as well.

For many years, starting with the seminal work of Shafranov (1957), Lüst and Schlüter (1957) and Grad and Rubin (1959), the bulk of the effort of the plasma physics community was directed towards studying static equilibria with axial symmetry. Not only are these equilibria of great practical value, but, in addition, their mathematical description is (relatively) straightforward. These equilibria are described by the Grad–Shafranov equation for the streamfunction, which is a semilinear elliptic equation with two profile functions describing the pressure and the toroidal magnetic field. Once appropriate boundary conditions have been chosen (these conditions might be either fixed- or free-boundary ones), the problem is accessible via standard methods. The existence theorem for the Grad–Shafranov problem has been established, and a number of highly efficient numerical codes dealing with this problem numerically are available. An important feature of the problem is that in static equilibrium the density remains undetermined and can be prescribed arbitrarily.

Only later did it became apparent that plasma flows play an important role in tokamak physics (see e.g. Zehrfeld and Green 1972; Maschke and Perrin 1980; Hameiri 1983; Bhattacharjee 1988; Chu *et al.* 1995), and their investigation began in earnest. At the same time, in the astrophysical context steady flows are of paramount importance (Sakurai 1990; Sauty 1994). Equations describing incompressible steady flows with symmetry were obtained in the 1950s by Chandrasekhar (1956) and Tkalich (1962). Their compressible counterparts were derived in the early 1960s (Soloviev 1967; Morozov and Soloviev 1980).

The equations governing plasma equilibria with flow have been discussed in both the fusion and astrophysical contexts by many authors, including Zehrfeld and Green (1972), Tsinganos (1981), Hameiri (1983) and Webb *et al.* (1994). In contrast to the static case, equilibria with flow are governed by a system of two coupled equations; one of them is a generalization of the Grad–Shafranov equation for the streamfunction, and the second is the Bernoulli equation, which is a highly nonlinear algebraic equation for the density. In general, these equations depend on six (or five) profile functions. Coupling of the Grad–Shafranov and Bernoulli equations has several important consequences. The most significant is that the resulting equation for the streamfunction is no longer elliptic. In this respect, the equilibrium problem closely resembles the classical problem of describing symmetric flows of a compressible fluid (Chaplygin 1904; von Mises 1958; Guderley 1962; Sedov 1965; Cole and Cook 1986). It is well known that describing *static plasma equilibria* is exactly equivalent to describing *incompressible fluid flows* (Lifschitz 1989). At the same time, the description of *steady plasma flows* is much more involved than the description of *compressible fluid flows* (Mitchner 1959; Grad 1960; Imai 1960; Chu 1962). Specifically, transonic fluid flows are characterized by a single dimensionless parameter (the Mach number), while plasma flows are characterized by two such parameters (the Mach and Alfvén–Mach numbers). As a result, for fluid flows there are only two distinct regimes (one elliptic and one hyperbolic), while for plasma flows there are five distinct regimes (three elliptic and two hyperbolic), and, in addition, there is the so-called (apparent) Alfvén singularity.

After decades of dedicated effort, the theory of transonic flows is more or less complete (although quite a number of important issues remain to be addressed, both analytically and numerically) (for an overview of the subject see e.g. Guderley 1962; Cole and Cook 1986). In particular, under certain assumptions, the existence theorem for transonic flows satisfying appropriate boundary conditions has been proved (Morawetz 1985; Kloucek and Necas 1990), and powerful numerical methods for finding these flows numerically have been developed (Jameson 1980, 1988; Bristeau *et al.* 1989). At the same time, no satisfactory theory is currently available for dealing with transonic plasma flows, and methods for describing them numerically often deal with the elliptic case, when the true nature of the problem is underemphasized (Gruber *et al.* 1985; Kerner and Tokuda 1987; Cooper and Hirshman 1987; Galkowski and Zelazny 1994), although in the astrophysical context transonic plasma flows of mixed elliptic–hyperbolic type have been considered by several investigators (see e.g. Sakurai 1990; Bogovalov 1994).

In view of the above, it is highly desirable to develop alternative ways for describing symmetric steady flows. In this paper we consider flows translationally invariant along the (vertical) z axis, and propose a new description of such flows based on a judicious choice of independent and dependent variables. Throughout the paper, we restrict ourselves to the case when the poloidal (in the (x, y) plane) components of the magnetic field and plasma velocity are parallel. Our approach is a generalization of the classical approach developed for compressible fluid flows in the 1950s (Sedov 1965), but seldomly (if ever) used since then. The idea is based on the observation that translationally symmetric flows do not depend explicitly on the physical coordinates in the (horizontal) (x, y) plane, which facilitates their description in terms of intrinsic (natural) coordinates associated with a given flow itself. As a particular set of intrinsic coordinates, we choose (ξ, η, ϑ), where ξ is a label of flux and streamsurfaces, η is the *total* pressure, and ϑ is the angle between $\nabla\xi$ (the direction of the horizontal magnetic and velocity field) and the x axis. In principle, any one of these variables can be chosen as a dependent variable, with the other two as independent ones. To start with, we assume that ϑ is our dependent variable, and derive a single nonlinear equation of mixed elliptic–hyperbolic type for $\vartheta(\xi, \eta)$. This equation depends on a single coefficient function absorbing all the information about the profile functions characterizing the flow. This equation is much easier to deal with than the conventional system of coupled equations for ξ and ρ considered as functions of (x, y). One of its great advantages is that the location of the elliptic and hyperbolic regions is known a priori. Needless to say, this is true only in the intrinsic coordinates (ξ, η), and, in order to find the location of these regions in the physical coordinates (x, y), we have to find the mapping $(\xi, \eta) \to (x, y)$. Next, we treat ξ as a dependent variable and derive a single equation for $\xi = \xi(\eta, \vartheta)$, which can be considered as a generalization of the conventional hodograph equilibrium equation for irrotational fluid flows. Under certain very restrictive assumptions, the equation for $\xi(\eta, \vartheta)$ is more convenient to use than the equation for $\vartheta(\xi, \eta)$, but, in general, the latter is much more useful than the former. We also derive an equation for $\eta(\xi, \vartheta)$, which can be considered as the generalization of the static equilibrium representation in inverse coordinates (Lao *et al.* 1981).

As usual, when intrinsic coordinates are used, it becomes very difficult to formulate the corresponding boundary conditions and to deal with them efficiently. However, our equations can be used for finding exact self-similar flows in the whole plane or in appropriate parts of it. The quest for exact self-similar fluid and plasma flows has a very long history, and many important results concerning such flows are currently available (Sedov 1959; Bickley 1964; Low 1975; Barenblatt 1979; Tsinganos 1981; Blandford and Payne 1982; Bogoyavlensky 1985; Tsinganos and Trussoni 1991; Bacciotti and Chiuderi 1992; Tsinganos and Surlantzis 1992; Tsinganos *et al.* 1993; Sauty and Tsinganos 1994; Sauty 1994; Del Zanna and Chiuderi 1996). There are many reasons why such solutions are important, the most obvious being that they can be used as approximations for more general non-self-similar solutions. We describe the most general self-similar solutions of our basic equilibrium equation for $\vartheta(\xi, \eta)$, and analyse certain cases when they can be found in a closed form. An alternative way of describing self-similar solutions in physical coordinates is developed in a separate paper (Goedbloed and Lifschitz 1997).

The paper is organized as follows. In Sec. 2 we present the standard coupled system of equilibrium equations written in the physical coordinates for translationally symmetric plasma flows. In Sec. 3 we give the three complementary descriptions of plasma flows in intrinsic coordinates. In Sec. 4 we analyse general properties of the equilibrium equations written in intrinsic coordinates. Next, we discuss conditions under which these equations have explicit solutions. We continue our discussion with three highly specialized types of flows, namely the so-called Prandtl–Meyer, spiral and Ringleb flows (Sec. 5). Next, we derive an ordinary differential equation describing the most general self-similar flows (Sec. 6). We analyse solutions of this equation in Sec. 7. In particular, we present certain cases when this equation can be solved in a closed form, and describe the corresponding flows in some detail. Our conclusions and future directions of research are outlined in Sec. 8.

Some of the results described below have been announced in Goedbloed and Lifschitz (1996).

2. Plasma flows in physical coordinates

As a starting point, in this section we present the general equilibrium equations for translationally invariant plasma flows written in physical coordinates.

The equations of ideal MHD describing steady plasma equilibria have the form

$$\nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{1}$$

$$\boldsymbol{\omega} \times \mathbf{v} - \frac{1}{\rho}\mathbf{j} \times \mathbf{B} + \nabla(\tfrac{1}{2}v^2) + \frac{1}{\rho}\nabla p = 0, \tag{2}$$

$$\mathbf{v} \cdot \nabla S = 0\,, \tag{3}$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \tag{4}$$

where $\rho, p, S, \mathbf{v}, \boldsymbol{\omega}, \mathbf{B}$ and $\mathbf{j}$ are the plasma density, pressure, entropy, velocity, vorticity, magnetic field and current density respectively. Equations (1)–(4) describe the conservation of mass, the force balance, the advection of the entropy and the induction law. The equation of state has the form

$$p = S\,\rho^{\gamma}\,, \tag{5}$$

where γ is the adiabaticity index.

We consider only equilibria that are translationally invariant along the z axis (the vertical axis). Accordingly, all the variables are independent of z. Below, subscripts p and z denote the projections of the corresponding vectors on the (x, y) plane and the z axis respectively.

An appropriate modification of the classical derivation shows that the dependent variables can be written in the form

$$\mathbf{B} = \psi' \mathbf{e}_z \times \nabla\xi + B_z \mathbf{e}_z\,, \qquad B_z = \frac{\psi' I' + \chi' \boldsymbol{\Omega}'}{\psi'^2 - \chi'^2/\rho}\,, \tag{6}$$

$$\mathbf{v} = \frac{\chi'}{\rho}\,\mathbf{e}_z \times \nabla\xi + v_z \mathbf{e}_z\,, \qquad v_z = \frac{\psi' \boldsymbol{\Omega}' + \chi' I'/\rho}{\psi'^2 - \chi'^2/\rho}\,, \tag{7}$$

$$\mathbf{j} \equiv \nabla \times \mathbf{B} = \nabla B_z \times \mathbf{e}_z + \nabla \cdot \left(\psi' \nabla\xi\right) \mathbf{e}_z\,, \tag{8}$$

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{v} = \nabla v_z \times \mathbf{e}_z + \nabla \cdot \left(\frac{\chi'}{\rho} \nabla\xi\right) \mathbf{e}_z\,, \tag{9}$$

where ξ is a label of magnetic and streamsurfaces, and $\psi, \chi, S, \boldsymbol{\Omega}$ and I are functions of ξ. Here ψ is the magnetic flux function, χ is the velocity streamfunction, S is the entropy, $\boldsymbol{\Omega}$ is the scalar potential of the electric field $\mathbf{E} = -\mathbf{v} \times \mathbf{B} = \nabla\boldsymbol{\Omega}$, and I is the scalar potential of the auxiliarly field $\mathbf{G} = \mathbf{B}_p \times \mathbf{B}_z - \rho \mathbf{v}_p \times \mathbf{v}_z = \nabla I$. Recall that we restrict ourselves to the case where $\mathbf{B}_p$ and $\mathbf{v}_p$ are parallel. We emphasize that, in contrast to the usual practice, we introduce an auxiliary label ξ of magnetic and streamsurfaces rather than just ψ or χ, because we want to be able to treat the horizontal magnetic field $\mathbf{B}_p$ and the velocity field $\mathbf{v}_p$ in a symmetric way (see also Morozov and Soloviev 1980; Del Zanna and Chiuderi 1996).

It is easy to verify that (1), (3), (4) and the vertical component of (2) are satisfied automatically, while the projections of the horizontal components of the force-balance equations orthogonal and parallel to the direction of streamlines give the so-called transverse equilibrium equation

$$\begin{aligned}\nabla \cdot \left[\left(\psi'^2 - \frac{\chi'^2}{\rho}\right) \nabla\xi\right] - \left(\psi'\psi'' - \frac{\chi'\chi''}{\rho}\right) |\nabla\xi|^2 + H'\rho - \frac{1}{\gamma - 1} S'\rho^\gamma \\ + \frac{\psi' I' + \chi' \boldsymbol{\Omega}'}{\psi'(\psi'^2 - \chi'^2/\rho)} \left(\frac{\psi' I' + \chi' \boldsymbol{\Omega}'}{\psi'}\right)' - \frac{(\psi' I' + \chi' \boldsymbol{\Omega}')^2 (\psi'\psi'' - \chi'\chi''/\rho)}{\psi'^2 (\psi'^2 - \chi'^2/\rho)^2} = 0,\end{aligned} \tag{10}$$

and the longitudinal equilibrium equation (the Bernoulli law)

$$\frac{1}{2} \frac{\chi'^2}{\rho^2} |\nabla\xi|^2 - H + \frac{\gamma}{\gamma - 1} S\rho^{\gamma-1} + \frac{1}{2} \frac{(\psi' I' + \chi' \boldsymbol{\Omega}')^2}{\psi'^2 (\psi'^2 - \chi'^2/\rho)^2} \frac{\chi'^2}{\rho^2} = 0\,, \tag{11}$$

where $H = \frac{1}{2}(B^2/B_z^2) v_p^2 + \gamma S \rho^{\gamma-1}/(\gamma - 1)$ is a function of ξ known as the Bernoulli head function, which is an appropriate modification of the Bernoulli function for an ideal gas. Thus, in general, plasma flows depend on six free functions $\psi, \chi, S, I, \boldsymbol{\Omega}$ and H. The number of free functions can be reduced by one if either ψ or χ is chosen as a label on flux and streamsurfaces.

Note that the functions $I(\xi), \boldsymbol{\Omega}(\xi)$ and $H(\xi)$ are related to the commonly used functions $\tilde{I}(\psi), \tilde{\boldsymbol{\Omega}}(\psi)$ and $\tilde{H}(\psi)$ (see e.g. Goedbloed and Lifschitz 1997) by

$$\tilde{I}(\psi) = \frac{I'(\xi)}{\psi'(\xi)}, \quad \tilde{\boldsymbol{\Omega}}(\psi) = \frac{\boldsymbol{\Omega}'(\xi)}{\psi'(\xi)}, \quad \tilde{H}(\psi) = H(\xi) - \frac{1}{2} \left[\frac{\boldsymbol{\Omega}'(\xi)}{\psi'(\xi)}\right]^2.$$

It is shown in Goedbloed and Lifschitz (1997) that (10) and (11) are the Euler–

Lagrange equations for the Lagrangian

$$\mathscr{L} = \int \left[\frac{1}{2} \left(\psi'^2 - \frac{\chi'^2}{\rho} \right) |\nabla\xi|^2 - H\rho + \frac{1}{\gamma - 1} S\rho^\gamma - \frac{1}{2} \frac{(\psi' I' + \chi'\Omega')^2}{\psi'^2(\psi'^2 - \chi'^2/\rho)} \right] dx\, dy\,. \quad (12)$$

A similar variational formulation is given by Rosso and Pelletier (1994). Alternatively, the equilibrium equations can be described by the Euler–Lagrange equations for a constrained Hamiltonian problem (see e.g. Almaguer *et al.* 1988). In terms of the physical variables, the Lagrangian $\mathscr{L}$ can be written as

$$\mathscr{L} = \int (B_p^2 - \rho v_p^2 - \eta)\, dx\, dy\,, \quad (13)$$

where $\eta = p + \frac{1}{2}B^2$ is the total pressure.

One of the main advantages of writing the equilibrium equations in terms of ξ (rather than ψ or χ) and ρ is the possibility of describing the very important limiting cases of plasma equilibria with vertical flow ($\chi' = 0$) and compressible fluid flows with vertical magnetic field ($\psi' = 0$) in a very natural way.

For plasma equilibria with vertical flow, (10) and (11) assume the form

$$\nabla \cdot (\psi'^2\, \nabla\xi) - \psi'\psi''|\nabla\xi|^2 + H'\rho - \frac{1}{\gamma - 1} S'\rho^\gamma + \frac{I'I''}{\psi'^2} - \frac{\psi'' I'^2}{\psi'^3} = 0, \quad (14)$$

$$-H + \frac{\gamma}{\gamma - 1} S\rho^{\gamma - 1} = 0\,. \quad (15)$$

For fluid flows with vertical magnetic field, (10) and (11) can be written as

$$-\nabla \cdot \left(\frac{\chi'^2}{\rho}\, \nabla\xi \right) + \frac{\chi'\chi''}{\rho}|\nabla\xi|^2 + \bar{H}'\rho - \frac{1}{\gamma - 1} S'\rho^\gamma - \frac{1}{2} \left(\frac{\Omega'^2}{\chi'^2} \right)' \rho^2 = 0\,, \quad (16)$$

$$\frac{1}{2} \frac{\chi'^2}{\rho^2} |\nabla\xi|^2 - \bar{H} + \frac{\gamma}{\gamma - 1} S\rho^{\gamma - 1} + \frac{\Omega'^2}{\chi'^2}\rho = 0\,, \quad (17)$$

where $\bar{H} = H - I'\Omega'/(\psi'\chi')$ is a renormalized Bernoulli function.

Equations (14), (15) and (16), (17) also allow a variational formulation.

3. Intrinsic descriptions of plasma flows

In this section we rewrite the general equilibrium equations for translationally invariant plasma flows in a new form. In contrast to the standard approach, here we rewrite the equilibrium equations in terms of natural coordinates associated with the equilibrium itself. Inspection of (10) and (11) shows that plasma flows can be characterized by the intrinsic variables ξ, η and ϑ, where ξ is a label of flux and streamsurfaces, η is the total pressure, and ϑ is the angle between the horizontal magnetic and velocity fields and the x axis. Under broad assumptions of invertibility of the corresponding mappings, we can use any of the three variables as dependent and take the other two as independent. Thus we have to consider three possibilities. First, we choose ϑ as a dependent variable, and consider ξ and η as independent variables. The resulting equation for $\vartheta(\xi, \eta)$ is a very convenient tool for studying plasma flows. In particular, it allows one to study the ellipticity–hyperbolicity transitions in a very natural way (see Sec. 4) as well as to analyse self-similar plasma flows (see Secs 6 and 7).

Next, we assume that ξ is our dependent variable, treat η and ϑ as independent variables, and obtain the so-called hodograph equation for $\xi(\eta, \vartheta)$. The hodograph equation, originally introduced by Chaplygin (1904) in his seminal work, has been extensively used for studying plane irrotational motions of ordinary fluids (von Mises 1958; Guderley 1962; Lighthill 1964). Its MHD analogues have been discussed by several investigators (Imai 1960; Sears 1960; Seebass 1961, Webb *et al.* 1994). The hodograph equation is a powerful, albeit delicate, tool for studying irrotational compressible flows, since it allows one to write the equilibrium conditions as a single, linear equation in the hodograph (velocity) plane. It is shown below that the hodograph equation can be derived for *rotational, three-dimensional* flows of magnetized plasmas as well. The resulting hodograph equation is no longer linear (in general). However, under certain (restrictive) conditions, it can be linearized and effectively solved. In Sec. 5 we give two examples of the corresponding solutions: the MHD analogue of the spiral flow in the plane and the classical Ringleb flow around a semi-infinite plate.

Finally, for the sake of completeness, we derive the equilibrium equation for $\eta(\vartheta, \xi)$.

Needless to say, the main difficulty in using the intrinsic equations stems from the need to invert coordinates in order to represent the flow in the physical plane. Whether or not such an inversion is possible depends on the details of the solution.

3.1. *The equation for $\vartheta(\xi, \eta)$*

First, we choose the label function ξ and the total pressure η as independent variables, and consider ϑ as a dependent one. We emphasize that this choice has certain limitations: it tacitly implies that level lines of ξ (and consequently of ψ and χ) are not closed. Although this looks like a serious limitation, in reality it is not, since in many cases flows with closed streamlines can be obtained by reflection of flows with open streamlines.

We introduce a horizontal vector field parallel to both $\mathbf{B}_p$ and $\mathbf{v}_p$:

$$\mathbf{Q} = \mathbf{e}_z \times \nabla\xi, \qquad \mathbf{B}_p = \psi'\mathbf{Q}, \qquad \mathbf{v}_p = \frac{\chi'}{\rho}\mathbf{Q}, \tag{18}$$

and use polar coordinates in the (Q_x, Q_y) plane to write the field $\mathbf{Q}$ in the form

$$Q_x = Q\cos\vartheta, \qquad Q_y = Q\sin\vartheta, \tag{19}$$

where ϑ is the angle between $\mathbf{Q}$ (as well as $\mathbf{B}_p$ and $\mathbf{v}_p$) and the x axis. Besides, we introduce the coefficient function $U(\xi, \eta)$ of the form

$$U(\xi, \eta) = \left(\psi'^2 - \frac{\chi'^2}{\rho}\right)|\nabla\xi| = \left(\psi'^2 - \frac{\chi'^2}{\rho}\right)Q = \frac{B_p^2 - \rho v_p^2}{Q}. \tag{20}$$

Next, we write the horizontal force-balance condition as

$$\nabla(\tfrac{1}{2}B_p^2) - \rho\nabla(\tfrac{1}{2}v_p^2) = (\psi' j_z - \chi'\omega_z)\nabla\xi + \nabla\eta. \tag{21}$$

The ξ and η components of (21) are

$$\frac{\partial}{\partial\xi}(\tfrac{1}{2}UQ) + \frac{\chi'^2}{2\rho^2}Q\frac{\partial\rho}{\partial\xi} = \psi' j_z - \chi'\omega_z. \tag{22}$$

$$Q\frac{\partial U}{\partial\eta} = 1. \tag{23}$$

In order to obtain a closed system of equations we have to express x and y in terms of ξ and η, or equivalently to find the expressions for the derivatives $\partial x/\partial\xi$, $\partial y/\partial\xi$, $\partial x/\partial\eta$ and $\partial y/\partial\eta$.

First, we note that

$$\begin{pmatrix} \dfrac{\partial\xi}{\partial x} & \dfrac{\partial\eta}{\partial x} \\ \dfrac{\partial\xi}{\partial y} & \dfrac{\partial\eta}{\partial y} \end{pmatrix} = \frac{1}{J}\begin{pmatrix} \dfrac{\partial y}{\partial\eta} & -\dfrac{\partial y}{\partial\xi} \\ -\dfrac{\partial x}{\partial\eta} & \dfrac{\partial x}{\partial\xi} \end{pmatrix}, \tag{24}$$

where J is the Jacobian of the transformation $(\xi,\eta)\to(x,y)$, i.e. $J=\partial(x,y)/\partial(\xi,\eta)$. Equation (24) shows that

$$(Q_x, Q_y) = \frac{1}{J}\left(\frac{\partial x}{\partial\eta}, \frac{\partial y}{\partial\eta}\right). \tag{25}$$

Equations (6) and (7) give similar expressions for v_x, v_y, B_x and B_y. Using (25), we can obtain the first pair of equations for the derivatives $\partial x/\partial\xi$, $\partial y/\partial\xi$, $\partial x/\partial\eta$ and $\partial y/\partial\eta$. These equations are purely geometrical in nature; they express the fact that $\partial\xi/\partial\xi = 1$ and $\partial\xi/\partial\eta = 0$ and can be written as

$$Q_y\frac{\partial x}{\partial\xi} - Q_x\frac{\partial y}{\partial\xi} = 1, \tag{26}$$

$$Q_y\frac{\partial x}{\partial\eta} - Q_x\frac{\partial y}{\partial\eta} = 0, \tag{27}$$

or equivalently, by using (19) and (23),

$$\sin\vartheta\,\frac{\partial x}{\partial\xi} - \cos\vartheta\,\frac{\partial y}{\partial\xi} = \frac{\partial U}{\partial\eta}, \tag{28}$$

$$\sin\vartheta\,\frac{\partial x}{\partial\eta} - \cos\vartheta\,\frac{\partial y}{\partial\eta} = 0. \tag{29}$$

Accordingly, we need two more equations for the derivatives in question. In order to use the equilibrium equation for this purpose we need to know the expressions for the vertical components of the vorticity and current. We have the following expressions for the contra- and covariant components of $\mathbf{Q}$:

$$Q^\xi = 0, \qquad Q^\eta = \frac{1}{J}, \qquad Q_\xi = \frac{g_{\xi\eta}}{J}, \qquad Q_\eta = \frac{g_{\eta\eta}}{J}, \tag{30}$$

where $g_{\xi\eta}$ and $g_{\eta\eta}$ are the elements of the metric tensor in (ξ,η) coordinates. The contra- and covariant components of $\mathbf{v}_p$ and $\mathbf{B}_p$ are proportional to those of $\mathbf{Q}$, with coefficients χ'/ρ and ψ' respectively. The elements of the metric tensor can be written as

$$g_{\xi\eta} = \frac{\partial x}{\partial\xi}\frac{\partial x}{\partial\eta} + \frac{\partial y}{\partial\xi}\frac{\partial y}{\partial\eta}, \qquad g_{\eta\eta} = \left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2. \tag{31}$$

Using the expressions for Q_x and Q_y we represent these coefficients in the form

$$g_{\xi\eta} = J\left(Q_x \frac{\partial x}{\partial \xi} + Q_y \frac{\partial y}{\partial \xi}\right), \qquad g_{\eta\eta} = J^2 Q^2. \tag{32}$$

Exploiting (25) and (31), we get the following expression for ω_z:

$$\begin{aligned}\omega_z &= \frac{1}{J}\frac{\partial}{\partial \xi}\left(\frac{\chi' g_{\eta\eta}}{J\rho}\right) - \frac{1}{J}\frac{\partial}{\partial \eta}\left(\frac{\chi' g_{\xi\eta}}{J\rho}\right) \\ &= \frac{\chi'}{\rho}\frac{\partial}{\partial \xi}(\tfrac{1}{2}Q^2) + \frac{\partial}{\partial \xi}\left(\frac{\chi'}{\rho}\right)Q^2 \\ &\quad - \frac{1}{J}\frac{\partial}{\partial \eta}\left(\frac{\chi' Q_x}{\rho}\right)\frac{\partial x}{\partial \xi} - \frac{1}{J}\frac{\partial}{\partial \eta}\left(\frac{\chi' Q_y}{\rho}\right)\frac{\partial y}{\partial \xi}.\end{aligned} \tag{33}$$

Similarly, for j_z,

$$\begin{aligned}j_z &= \frac{1}{J}\frac{\partial}{\partial \xi}\left(\frac{\psi' g_{\eta\eta}}{J}\right) - \frac{1}{J}\frac{\partial}{\partial \eta}\left(\frac{\psi' g_{\chi\eta}}{J}\right) \\ &= \psi'\frac{\partial}{\partial \xi}(\tfrac{1}{2}Q^2) + \psi'' Q^2 - \frac{1}{J}\frac{\partial(\psi' Q_x)}{\partial \eta}\frac{\partial x}{\partial \xi} \\ &\quad - \frac{1}{J}\frac{\partial(\psi' Q_y)}{\partial \eta}\frac{\partial y}{\partial \xi}.\end{aligned} \tag{34}$$

Substituting these expressions into the equilibrium equation (22) and using the notation (19) and (20), we obtain the relation for $\partial x/\partial \xi$ and $\partial y/\partial \xi$ that we sought:

$$\frac{\partial}{\partial \eta}(U\cos\vartheta)\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial \eta}(U\sin\vartheta)\frac{\partial y}{\partial \xi} = 0. \tag{35}$$

Exploiting (23), (26) and (35), we obtain

$$\left(\frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \xi}\right) = \left(\frac{\partial}{\partial \eta}(U\sin\vartheta), -\frac{\partial}{\partial \eta}(U\cos\vartheta)\right). \tag{36}$$

Now we have to derive an additional equation for $\partial x/\partial \eta$ and $\partial y/\partial \eta$. To this end, we differentiate (29) with respect to ξ, and use (36) to obtain

$$\cos\vartheta\frac{\partial x}{\partial \eta} + \sin\vartheta\frac{\partial y}{\partial \eta} = -\frac{\partial^2 U/\partial \eta^2 - U(\partial\vartheta/\partial\eta)^2}{\partial\vartheta/\partial\xi}. \tag{37}$$

Using (29), we obtain the following expressions for $\partial x/\partial \eta$ and $\partial y/\partial \eta$:

$$\left(\frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \eta}\right) = -\frac{\partial^2 U/\partial \eta^2 - U(\partial\vartheta/\partial\eta)^2}{\partial\vartheta/\partial\xi}(\cos\vartheta, \sin\vartheta). \tag{38}$$

From now on, we consider the angle ϑ between the vector $\mathbf{Q}$ and the x axis as our *dependent* variable. In order to derive an equation for ϑ more easily, we introduce complex coordinate $\zeta = x + iy$ in the (x, y) plane. In complex coordinates, (36) and (38) can be written as

$$\frac{\partial \zeta}{\partial \xi} = -i\frac{\partial}{\partial \eta}[U\exp(i\vartheta)] = \left(-i\frac{\partial U}{\partial \eta} + U\frac{\partial \vartheta}{\partial \eta}\right)\exp(i\vartheta), \tag{39}$$

$$\frac{\partial \zeta}{\partial \eta} = -\frac{\partial^2 U/\partial \eta^2 - U(\partial\vartheta/\partial\eta)^2}{\partial\vartheta/\partial\xi}\exp(i\vartheta), \tag{40}$$

or equivalently

$$d\zeta = \exp(i\vartheta)\left[\left(-i\frac{\partial U}{\partial\eta} + U\frac{\partial\vartheta}{\partial\eta}\right)d\xi - \frac{\partial^2 U/\partial\eta^2 - U\,(\partial\vartheta/\partial\eta)^2}{\partial\vartheta/\partial\xi}\,d\eta\right]. \tag{41}$$

The compatibility condition $\partial^2\zeta/\partial\xi\,\partial\eta = \partial^2\zeta/\partial\eta\,\partial\xi$ yields the equation for ϑ that we sought:

$$\frac{\partial}{\partial\xi}\left[\frac{\partial^2 U/\partial\eta^2 - U\,(\partial\vartheta/\partial\eta)^2}{\partial\vartheta/\partial\xi}\right] + \frac{1}{U}\frac{\partial}{\partial\eta}\left(U^2\frac{\partial\vartheta}{\partial\eta}\right) = 0. \tag{42}$$

Explicitly, (42) can be written as

$$\begin{aligned} -\left[\frac{\partial^2 U}{\partial\eta^2} - U\left(\frac{\partial\vartheta}{\partial\eta}\right)^2\right]\frac{\partial^2\vartheta}{\partial\xi^2} - 2U\frac{\partial\vartheta}{\partial\xi}\frac{\partial\vartheta}{\partial\eta}\frac{\partial^2\vartheta}{\partial\xi\partial\eta} + U\left(\frac{\partial\vartheta}{\partial\xi}\right)^2\frac{\partial^2\vartheta}{\partial\eta^2} \\ + \frac{\partial^3 U}{\partial\xi\partial\eta^2}\frac{\partial\vartheta}{\partial\xi} - \frac{\partial U}{\partial\xi}\frac{\partial\vartheta}{\partial\xi}\left(\frac{\partial\vartheta}{\partial\eta}\right)^2 + 2\frac{\partial U}{\partial\eta}\left(\frac{\partial\vartheta}{\partial\xi}\right)^2\frac{\partial\vartheta}{\partial\eta} = 0. \end{aligned} \tag{43}$$

Equation (42) augmented with (20) defining the profile U is the key equation of two-dimensional steady magnetohydrodynamics. When supplied with appropriate boundary conditions, it describes a vast variety of MHD flows. It is remarkable that (42) describing magnetohydrodynamic flows is *identical* to the basic equation describing fluid flows as derived by Sedov (1965), although the meaning of the profile function U for these equations is quite different.

If a solution of (42) is known, the corresponding $\zeta = x + iy$ can be determined by integration of (41). The Jacobian of the transformation $(\xi,\eta) \to (x,y)$ can be written as

$$J = \frac{(x,y)}{(\xi,\eta)} = -\frac{\partial U}{\partial\eta}\left[\frac{\partial^2 U/\partial\eta^2 - U\,(\partial\vartheta/\partial\eta)^2}{\partial\vartheta/\partial\xi}\right]. \tag{44}$$

This transformation is non-invertible only on the so-called limit lines (see e.g. Guderley 1962) where

$$\text{either}\quad \frac{\partial^2 U}{\partial\eta^2} = U\left(\frac{\partial\vartheta}{\partial\eta}\right)^2 \quad\text{or}\quad \frac{\partial\vartheta}{\partial\xi} = \infty. \tag{45}$$

We emphasize that, in general, $\partial U/\partial\eta \neq 0$ by virtue of (23).

3.2. The hodograph equation for $\xi(\eta,\vartheta)$

Since we have already done most of the work required to write the equilibrium equation in intrinsic coordinates while deriving (42), we can now do the change of variables directly in this equation rather than in the system of coupled equations (10) and (11). As far as (42) is concerned, the hodograph transformation is simply an interchange between the variables ϑ and ξ. Expressing partial derivatives of ϑ with respect to ξ and η in terms of partial derivatives of ξ with respect to η and ϑ and substituting them into (43), we obtain the following equilibrium equation for $\xi(\eta,\vartheta)$:

$$U\frac{\partial^2\xi}{\partial\eta^2} - \frac{\partial^2 U}{\partial\eta^2}\frac{\partial^2\xi}{\partial\vartheta^2} - \frac{\partial^3 U}{\partial\xi\partial\eta^2}\left(\frac{\partial\xi}{\partial\vartheta}\right)^2 + \frac{\partial U}{\partial\xi}\left(\frac{\partial\xi}{\partial\eta}\right)^2 + 2\frac{\partial U}{\partial\eta}\frac{\partial\xi}{\partial\eta} = 0, \tag{46}$$

where $U(\xi,\eta)$ is given by (20).

In general, (46) is a quasilinear equation of mixed elliptic–hyperbolic type. The mapping $(\eta, \vartheta) \to (x, y)$ is given by the relation

$$d\zeta = -\exp(i\vartheta)\left[\left(i\frac{\partial U}{\partial \eta}\frac{\partial \xi}{\partial \eta} + \frac{\partial^2 U}{\partial \eta^2}\frac{\partial \xi}{\partial \vartheta}\right)d\eta + \left(i\frac{\partial U}{\partial \eta}\frac{\partial \xi}{\partial \vartheta} + U\frac{\partial \xi}{\partial \eta}\right)d\vartheta\right]. \tag{47}$$

The corresponding Jacobian has the form

$$J = \frac{(x, y)}{(\eta, \vartheta)} = \frac{\partial U}{\partial \eta}\left[\frac{\partial^2 U}{\partial \eta^2}\left(\frac{\partial \xi}{\partial \vartheta}\right)^2 - U\left(\frac{\partial \xi}{\partial \eta}\right)^2\right]. \tag{48}$$

Thus the limit lines are determined by the condition

$$\frac{\partial^2 U}{\partial \eta^2}\left(\frac{\partial \xi}{\partial \vartheta}\right)^2 = U\left(\frac{\partial \xi}{\partial \eta}\right)^2. \tag{49}$$

3.3. The equation for $\eta(\vartheta, \xi)$

For the sake of completeness, we also write the equilibrium equation using η as a dependent variable, and ϑ and ξ as independent ones. Expressing partial derivatives of ξ with respect to η and ϑ in terms of partial derivatives of η with respect to ξ and ϑ and substituting these expressions into (46), we obtain the following equilibrium equation for $\eta(\vartheta, \xi)$:

$$\begin{aligned}\frac{\partial^2 U}{\partial \eta^2}\left(\frac{\partial \eta}{\partial \xi}\right)^2\frac{\partial^2 \eta}{\partial \vartheta^2} - 2\frac{\partial^2 U}{\partial \eta^2}\frac{\partial \eta}{\partial \xi}\frac{\partial \eta}{\partial \vartheta}\frac{\partial^2 \eta}{\partial \xi \partial \vartheta} + \left[\frac{\partial^2 U}{\partial \eta^2}\left(\frac{\partial \eta}{\partial \vartheta}\right)^2 - U\right]\frac{\partial^2 \eta}{\partial \xi^2}&\\ - \frac{\partial^3 U}{\partial \xi \partial \eta^2}\frac{\partial \eta}{\partial \xi}\left(\frac{\partial \eta}{\partial \vartheta}\right)^2 + \frac{\partial U}{\partial \xi}\frac{\partial \eta}{\partial \xi} + 2\frac{\partial U}{\partial \eta}\left(\frac{\partial \eta}{\partial \xi}\right)^2 = 0.&\end{aligned} \tag{50}$$

As before, this is a mixed-type equation. The mapping $(\vartheta, \xi) \to (x, y)$ is given by the relation

$$d\zeta = \exp(i\vartheta)\left[\frac{(\partial^2 U/\partial \eta^2)(\partial \eta/\partial \vartheta)^2 - U}{\partial \eta/\partial \xi}d\vartheta + \left(-i\frac{\partial U}{\partial \eta} + \frac{\partial^2 U}{\partial \eta^2}\frac{\partial \eta}{\partial \vartheta}\right)d\xi\right]. \tag{51}$$

The Jacobian of this mapping has the form

$$J = \frac{(x, y)}{(\vartheta, \xi)} = -\frac{\partial U}{\partial \eta}\left[\frac{(\partial^2 U/\partial \eta^2)\,(\partial \eta/\partial \vartheta)^2 - U}{\partial \eta/\partial \xi}\right], \tag{52}$$

and the limit lines where

$$\text{either}\quad \frac{\partial^2 U}{\partial \eta^2}\left(\frac{\partial \eta}{\partial \vartheta}\right)^2 = U \quad\text{or}\quad \frac{\partial \eta}{\partial \xi} = \infty. \tag{53}$$

Comparison of (43), (46) and (50) shows that, depending on the physical problem at hand, it might be beneficial to use (43), (46) or (50). The main advantage of (43) is that its regions of ellipticity and hyperbolicity are known a priori (see below). On the other hand, in the special case where the profile function U is independent of ξ, $U(\xi, \eta) = f(\eta)$, or, more generally, when U is separable, $U(\xi, \eta) = e(\xi)f(\eta)$, (46) becomes linear while (43) remains nonlinear.

4. Properties of the equilibrium equations

4.1. *The profile function* $U(\xi, \eta)$

In order to deal with (42) in an efficient way, it is necessary to express the coefficient function U in terms of ξ and η. Unfortunately, a simple expression for $U(\xi, \eta)$ is not available; moreover, in general, $U(\xi, \eta)$ is not a single-valued function of its arguments. Nevertheless, a parametric representation of both η and U in terms of ξ and an additional auxiliary variable can easily be found. Whenever possible, we shall use the squared horizontal Alfvén Mach number, M^2, defined by the relation

$$M^2 = \frac{v_p^2}{v_{A,p}^2} = \frac{\chi'^2}{\psi'^2 \rho}, \tag{54}$$

where $v_{A,p}$ is the horizontal Alfvén speed, as our auxiliary variable. Following Goedbloed and Lifschitz (1997), we introduce the profiles $\Pi_i(\xi), i = 1, 2, 3$, of the form

$$\Pi_1 = \frac{\chi'^2}{\psi'^2} H, \qquad \Pi_2 = \frac{\gamma}{\gamma - 1} \left(\frac{\chi'^2}{\psi'^2}\right)^{\gamma} S, \qquad \Pi_3 = \frac{1}{2} \left(\frac{\psi' I' + \chi' \Omega'}{\psi'^2}\right)^2. \tag{55}$$

We use the equation of state and the Bernoulli law to represent η, B_p^2 and $U = (1 - M^2)\psi' B_p$ as functions of ξ and M^2:

$$\eta = \frac{\Pi_1(\xi)}{M^4} - \frac{\Pi_2(\xi)}{M^{2(\gamma+1)}} + \frac{\gamma - 1}{\gamma} \frac{\Pi_2(\xi)}{M^{2\gamma}}, \tag{56}$$

$$B_p^2 = 2 \left[\frac{\Pi_1(\xi)}{M^4} - \frac{\Pi_2(\xi)}{M^{2(\gamma+1)}} - \frac{\Pi_3(\xi)}{(1 - M^2)^2}\right], \tag{57}$$

$$U = (1 - M^2) \left\{2\psi'^2(\xi) \left[\frac{\Pi_1(\xi)}{M^4} - \frac{\Pi_2(\xi)}{M^{2(\gamma+1)}} - \frac{\Pi_3(\xi)}{(1 - M^2)^2}\right]\right\}^{1/2}. \tag{58}$$

Equations (56)–(58) define B_p^2 and U parametrically as functions of ξ and η.

On a given flux and streamline ξ = const, the shape of the functions $\eta(M^2)$, $B_p^2(M^2)$ and $U(M^2)$ depends on the non-dimensional ratios $A = \Pi_2/\Pi_1 > 0$ and $B = \Pi_3/\Pi_1 > 0$. Since both η and B_p^2 have to be positive, the values of A and B for a particular streamline ought to be such that for certain M^2 both $\eta > 0$ and $B_p > 0$. In Fig. 1 we show the division of the first quadrant in the (A, B) plane into the regions corresponding to topologically different behaviour of $\eta(M^2)$ and $B_p^2(M^2)$. For the purpose of illustration, here and below we assume that $\gamma = 2$. We emphasize that this assumption is not related in any sense to the fact that for fluid flows in vertical magnetic field the value $\gamma = 2$ has a very special meaning (see below). To construct Fig. 1, we note the obvious inequality $\eta \geqslant \frac{1}{2}B_p^2$ and study the conditions guaranteeing that $\max_{0 \leqslant M^2 < \infty} B_p^2(M^2) \geqslant 0$. Since the value $M^2 = 1$ is singular, it is convenient to study the slow (sub-Alfvénic) interval $0 \leqslant M^2 < 1$ and the fast (super-Alfvénic) interval $1 < M^2 < \infty$ separately. Accordingly, we define $m_s = \max_{0 \leqslant M^2 < 1} B_p^2(M^2)$ and $m_f = \max_{1 < M^2 < \infty} B_p^2(M^2)$. It is easy to show that $m_s > 0$ if and only if

$$A < 1 \quad \text{and} \quad B < f(A) \equiv \frac{4}{27} \frac{(1 - A)^3}{A^2}, \tag{59}$$

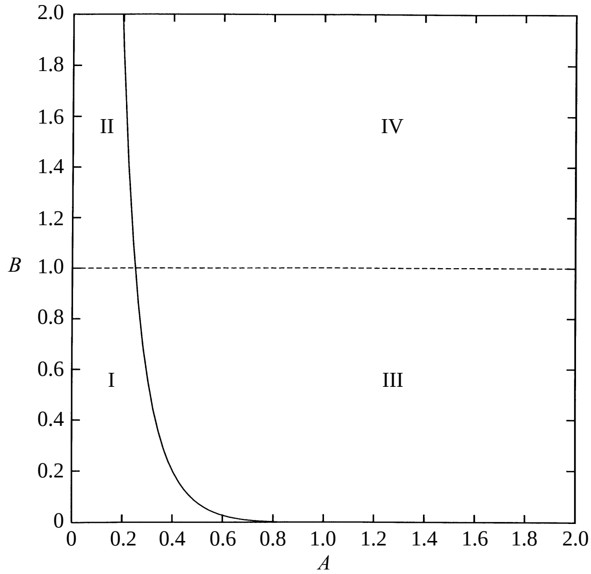


Figure 1. The positive quadrant in the (A, B) plane divided into four regions according to the behaviour of the functions $\eta(M^2)$ and $B_p^2(M^2)$. Both η and B_p^2 are positive on a sub-Alfvénic interval $M_1^2(A, B) < M^2 < M_2^2(A, B)$ and a super-Alfvénic interval $M_3^2(A, B) < M^2 < \infty$ (for points (A, B) belonging to the region I); only on a super-Alfvénic interval $M_3^2(A, B) < M^2 < \infty$ (for points (A, B) belonging to the region II); only on a sub-Alfvénic interval $M_1^2(A, B) < M^2 < M_2^2(A, B)$ (for points (A, B) belonging to the region III). There are no intervals where η and B_p^2 are simultaneously positive for points (A, B) belonging to the region IV.

while $m_f > 0$ if and only if $B < 1$. The curves $B = f(A)$, $0 < A < 1$, and $B = 1$ divide the first quadrant into four regions, which are shown in Fig. 1. We concentrate on studying the most interesting region I, where both m_s and m_f are positive, so that the corresponding B_p^2 has two local maxima at $M^2 = M_s^2$ and $M^2 = M_f^2$, while η has one maximum at $M^2 = M_c^2$, where M_c^2, M_s^2 and M_f^2 are the squared cusp, slow and fast Mach numbers ordered according to the familiar sequence of inequalities $M_c^2 \leqslant M_s^2 \leqslant 1 \leqslant M_f^2$. It is easy to show that B_p^2 is positive on two disjoint intervals $M_1^2 \leqslant M^2 \leqslant M_2^2$ (the sub-Alfvénic interval) and $M_3^2 \leqslant M^2 < \infty$ (the super-Alfvénic interval).

For a triple $\Pi_1 = 1$, $\Pi_2 = A$, $\Pi_3 = B$, where $(A, B) = (0.7, 0.001)$ is a typical point belonging to the region I in Fig. 1, we construct the single-valued functions $\eta(X), B_p^2(X)$ and $|U(X)|$, where $X = M^{-2}$, and the multivalued function $|U(\eta)|$, and show them in Figs 2(a, b).

4.2. The limiting profile functions $U(\xi, \eta)$

Equations (56) and (58) are quite appealing, but unfortunately it is difficult to use them in the two important limiting cases mentioned in Sec. 2, namely for plasma equilibria with vertical flow ($\psi' = 0$, $M^2 = 0$) and for compressible fluid flows with vertical magnetic field ($\xi' = 0$, $M^2 = \infty$). These cases have to be treated separately.

First we consider equilibria with vertical flow. For such equilibria, we use the

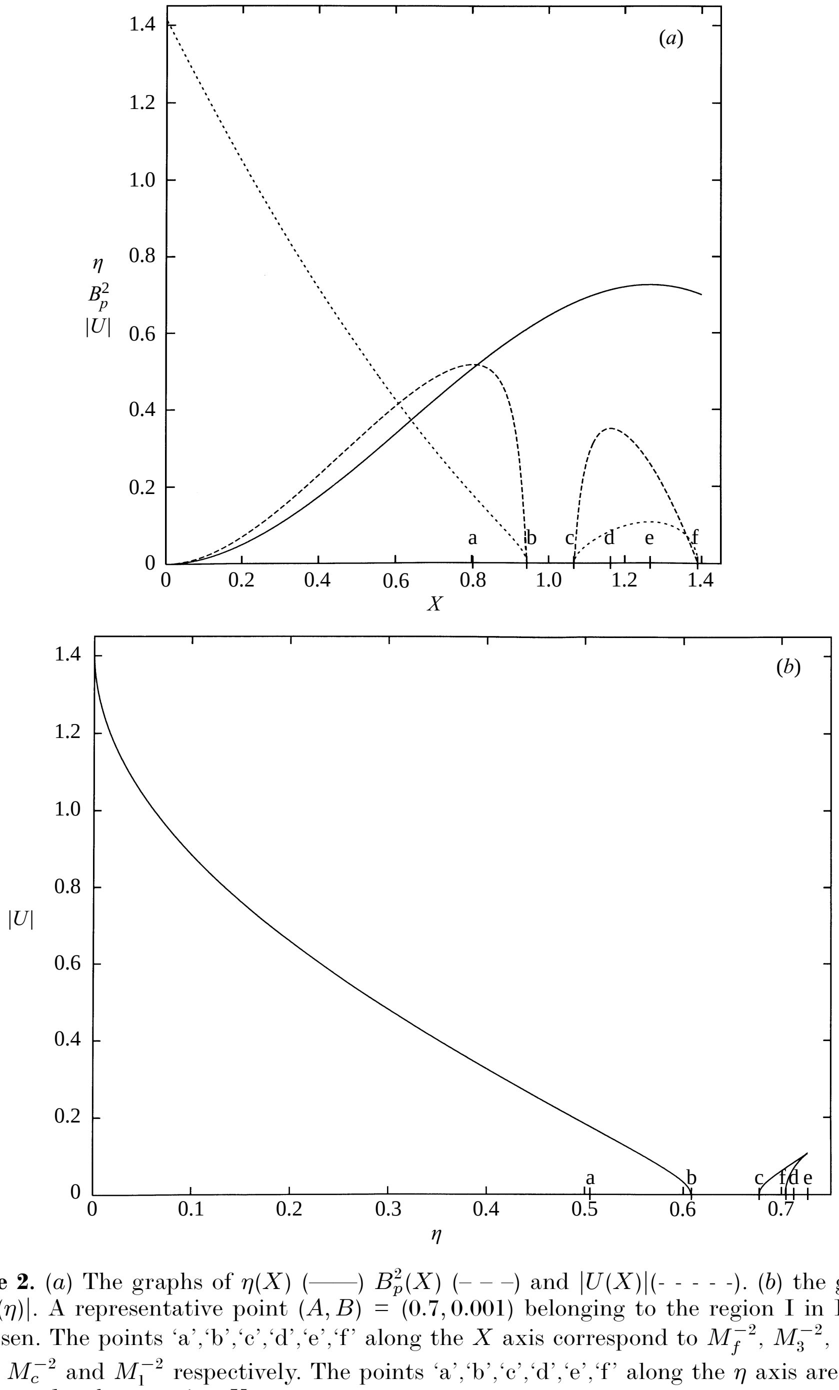


Figure 2. (a) The graphs of $\eta(X)$ (——) $B_p^2(X)$ (– – –) and $|U(X)|$(- - - - -). (b) the graph of $|U(\eta)|$. A representative point $(A, B) = (0.7, 0.001)$ belonging to the region I in Fig. 1 is chosen. The points 'a','b','c','d','e','f' along the X axis correspond to M_f^{-2}, M_3^{-2}, M_2^{-2}, M_s^{-2}, M_c^{-2} and M_1^{-2} respectively. The points 'a','b','c','d','e','f' along the η axis are their images under the mapping $X \to \eta$.

equation of state and the Bernoulli law to represent ρ and p as

$$\rho = \left(\frac{\gamma - 1}{\gamma}\right)^{1/(\gamma-1)} \left(\frac{H}{S}\right)^{1/(\gamma-1)}, \quad p = \left(\frac{\gamma - 1}{\gamma}\right)^{\gamma/(\gamma-1)} \left(\frac{H}{S}\right)^{1/(\gamma-1)} H. \tag{60}$$

These equations show that for plasma equilibria both the density ρ and the pressure p are functions of ξ only. Next, we use the definition of the total pressure $\eta = p+\frac{1}{2}B^2$, and write $U = \psi' B_p$ as

$$U = [2\psi'^2(\eta - p - \tfrac{1}{2}B_z^2)]^{1/2} = [2\psi'^2(\eta - \Lambda)]^{1/2}, \tag{61}$$

where

$$\Lambda = \left(\frac{\gamma-1}{\gamma}\right)^{\gamma/(\gamma-1)} \left(\frac{H}{S}\right)^{1/\gamma-1)} H + \frac{1}{2}\frac{I'^2}{\psi'^2} \tag{62}$$

is a function of ξ. It is clear that U is defined for $\eta > \Lambda$ only; it is a non-negative concave function of η.

Next we consider compressible fluid flows with vertical magnetic field. For such flows, η and $U = -\chi' v_p$ can be expressed in terms of ξ and $\bar{\rho} = [\gamma S/(\gamma-1]\bar{H})^{1/(\gamma-1)}\rho$ as follows:

$$\eta = p + \tfrac{1}{2}B_z^2 = \frac{\gamma-1}{\gamma}\Lambda_1\bar{\rho}^{\gamma} + \Lambda_2\bar{\rho}^2, \tag{63}$$

$$U = -\chi' v_p = -\left\{2\chi'^2\left(\frac{\gamma}{\gamma-1}\right)^{1/(\gamma-1)}\left(\frac{S}{\bar{H}}\right)^{1/(\gamma-1)}[\Lambda_1(1-\bar{\rho}^{\gamma-1}) - 2\Lambda_2\bar{\rho}]\right\}^{1/2}, \tag{64}$$

where

$$\left.\begin{aligned} \Lambda_1 &= \left(\frac{\gamma-1}{\gamma}\right)^{1/(\gamma-1)}\left(\frac{\bar{H}}{S}\right)^{1/(\gamma-1)}\bar{H}, \\ \Lambda_2 &= \frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)^{2/(\gamma-1)}\left(\frac{\bar{H}}{S}\right)^{2/(\gamma-1)}\frac{\Omega'^2}{\chi'^2}. \end{aligned}\right\} \tag{65}$$

The shape of the functions $\eta(\bar{\rho})$ and $U(\bar{\rho})$ is determined by the non-dimensional ratio $C = \Lambda_2/\Lambda_1$. For fixed C, the total pressure η is a monotonic function of the non-dimensional density $\bar{\rho}$. Accordingly, (63) is always invertible and U is a single-valued function of η.

Equation (63) can be inverted explicitly in two cases: (a) when non-magnetized fluids are considered, so that the vertical magnetic field vanishes and $\Lambda_2 = 0$, and (b) when $\gamma = 2$. The latter case was emphasized in the early literature on plasma flows (see e.g Mitchner 1959; Grad 1960). For non-magnetized fluids, (63) and (64) yield the following expression for U in terms of ξ and η:

$$U = -\left\{2\frac{\gamma}{\gamma-1}\chi'^2 S^{1/\gamma}\left[\left(\frac{\gamma-1}{\gamma}\Lambda_1\right)^{(\gamma-1)/\gamma} - \eta^{(\gamma-1)/\gamma}\right]\right\}^{1/2}. \tag{66}$$

It is interesting to note that the expressions (61) and (66) are equivalent (up to obvious changes) in the incompressible limit $\gamma = \infty$, which is a reflection of the well-known equivalence between static plasma equilibria and flows of incompressible fluid (see e.g. Lifschitz 1989).

4.3. *Transitions from ellipticity to hyperbolicity*

Equation (42) can be both elliptic and hyperbolic. It is easy to show that the type of this equation is determined by the following conditions:

$$U\frac{\partial^2 U}{\partial\eta^2} \quad \begin{cases} < 0 & \text{(elliptic)}, \\ = 0 & \text{(parabolic)}, \\ > 0 & \text{(hyperbolic)}. \end{cases} \tag{67}$$

The physical meaning of the transition conditions (67) becomes especially clear for the hodograph equation (46). In order to demonstrate that these conditions are equivalent to the standard transition conditions (see e.g. Goedbloed and Lifschitz 1997) we use (23), and rewrite the conditions (67) as

$$(1-M^2)\frac{\partial B_p^2/\partial M^2}{\partial\eta/\partial M^2}\quad\begin{cases} >0 & \text{(elliptic)},\\ =0 & \text{(parabolic)},\\ <0 & \text{(hyperbolic)}\end{cases} \tag{68}$$

Here, for convenience, we omit the negative factor $-\psi'^2/(2B_p^2)$ and use M^2 rather than η as a variable. It is clear that, on a given streamline, transitions from ellipticity to hyperbolicity occur, depending on M^2. Recall that, in general, B_p^2 has two local maxima at $M^2 = M_s^2$ and $M^2 = M_f^2$ while η has one maximum at $M^2 = M_c^2$. Accordingly, on increasing M^2, three elliptic and two hyperbolic regions are encountered:

$$\left.\begin{array}{c}\mathscr{E}_{ss}\colon\ M_1^2 < M^2 < M_c^2,\quad \mathscr{H}_s\colon\ M_c^2 < M^2 < M_s^2,\quad \mathscr{E}_s\colon\ M_s^2 < M^2 < M_2^2;\\ \mathscr{E}_f\colon\ M_3^2 < M^2 < M_f^2,\quad \mathscr{H}_f\colon\ M^2 > M_f^2,\end{array}\right\} \tag{69}$$

here $M_i^2, i = 1,2,3$, are the cutoff values of M^2 (cf. Fig. 2). We emphasize that the transitions occurring at $M = M_{s,f}$ and $M = M_c$ are very different in nature, because in the first case the function $U\,\partial^2U/\partial\eta^2$ changes its sign by passing through zero, while in the second case it does so by passing through infinity. It is clear that for equilibria with vertical flow, (42) is always elliptic (as might be expected). For flows with vertical magnetic field, this equation can be both elliptic and hyperbolic, but only regular transitions can occur. One of the main advantages of writing the equilibrium equation in natural coordinates is that the location of the domains of ellipticity and hyperbolicity can be determined in advance without solving this equation first. We emphasize that this is true only in natural coordinates. In order to describe these domains in (x, y) coordinates, one has to know the corresponding solution first.

In the degenerate case when the magnetic field is purely two-dimensional, $B_z = 0$, either $M_s^2 = M_a^2$ or $M_f^2 = M_a^2$, and the number of possible transitions decreases accordingly.

When (42) is hyperbolic, its characteristics are defined by the equation

$$\frac{\partial\vartheta}{\partial\xi}\,d\xi + \left[\pm\left(\frac{1}{U}\frac{\partial^2U}{\partial\eta^2}\right)^{1/2} + \frac{\partial\vartheta}{\partial\eta}\right]d\eta = 0. \tag{70}$$

They form two families according to the choice of either + or − sign. At a given point, there are two angles determining the deviation from the direction of $\mathbf{Q}$, which are the famous Mach angles. The following relations are satisfied on the characteristics:

$$d\vartheta = \mp\left(\frac{1}{U}\frac{\partial^2U}{\partial\eta^2}\right)^{1/2} d\eta = \frac{[(\partial^2U/\partial\eta^2)/U]^{1/2}\,\partial\vartheta/\partial\xi}{[(\partial^2U/\partial\eta^2)/U]^{1/2} \pm \partial\vartheta/\partial\eta}\,d\xi, \tag{71}$$

$$d\zeta = \exp(i\vartheta)\left[-i\frac{\partial U}{\partial\eta} \mp \left(U\frac{\partial^2U}{\partial\eta^2}\right)^{1/2}\right]d\xi. \tag{72}$$

It is clear that the limit lines given by (45) can be located only in the hyperbolic domain.

In the hodograph plane the characteristics are defined by the equation

$$d\xi \pm \left(\frac{1}{U}\frac{\partial^2 U}{\partial \eta^2}\right)^{1/2} d\eta = 0. \tag{73}$$

When the profile function U is separable, i.e. $U(\xi,\eta) = e(\xi)f(\eta)$, the form of the characteristics is independent of the choice of a particular solution ξ of the hodograph equation.

4.4. A variational principle for the intrinsic equilibrium equation

It was mentioned in Sec. 2 that the equilibrium equations (10) and (11) are the Euler–Lagrange equations for the Lagrangian (13). It is natural to expect that the intrinsic equilibrium equation (42) is the Euler–Lagrange equation for a certain Lagrangian too. Although for general profile functions $U(\xi,\eta)$ the corresponding Lagrangian is not available at present, it can be shown that for separable profile functions $U(\xi,\eta) = e(\xi)f(\eta)$ it can be chosen in the form

$$\begin{aligned}\mathscr{L} &= -\int U\left[\frac{\partial^2 U/\partial\eta^2 - U\,(\partial\vartheta/\partial\eta)^2}{\partial\vartheta/\partial\xi}\right] d\xi\, d\eta \\ &= \int \frac{U}{\partial U/\partial\eta} J\, d\xi\, d\eta = \int (B_p^2 - \rho v_p^2) J\, d\xi\, d\eta.\end{aligned} \tag{74}$$

4.5. Two simple geometrical results

Now we present some simple results concerning the general properties of solutions of (42). Since this equation *formally* coincides with its fluid counterpart, we can build upon the results for compressible fluid flows due to Nikol'ski, Taganov and Sedov (see Sedov 1965). It turns out, though, that the MHD case has some important distinctions.

First we consider level lines of $\vartheta = \vartheta' = \text{const}$ and analyse the behaviour of η along these lines. Let $\hat{\phi}$ be the angle between the level line and the poloidal velocity field at the point ζ_0. The tangent and the normal to this level line are denoted by ds and dn respectively. Following Sedov (1965), we write the relation between $d\eta/ds$ and $d\vartheta/dn$ as

$$\frac{d\eta}{ds} = \frac{(\partial U/\partial\eta)^2(\cos\hat{\phi})^2 - U(\partial^2 U/\partial\eta^2)(\sin\hat{\phi})^2}{(\partial U/\partial\eta)(\partial^2 U/\partial\eta^2)}\frac{d\vartheta}{dn}. \tag{75}$$

Using equation (23) we rewrite this relation in the form

$$\frac{d\eta}{ds} = \frac{(1-M^2)B_p^2[(\partial U/\partial\eta)^2(\cos\hat{\phi})^2 - U(\partial^2 U/\partial\eta^2)(\sin\hat{\phi})^2]}{U(\partial^2 U/\partial\eta^2)}\frac{d\vartheta}{dn}. \tag{76}$$

This equation clearly shows that if the level line under consideration belongs to an elliptic domain then η is a monotonic function of s. However, while for fluid flows we can conclude that $\eta \equiv p$ is an increasing function provided that the domain where $\vartheta > \vartheta'$ is on the right of the curve, for plasma flows it can be *both* an increasing and a decreasing function of s, depending whether $M^2 < 1$ or $M^2 > 1$. In both cases we can conclude that level lines of ϑ that belong to elliptic domains cannot be closed, since η has to be a single-valued function.

The behaviour of ϑ on level lines of η can be analysed in a similar way. Using the same notation as before, we can obtain the following relation between $d\vartheta/ds$ and $d\eta/dn$:

$$\frac{d\vartheta}{ds} = \frac{(\partial U/\partial \eta)^2(\cos\hat{\phi})^2 - U(\partial^2 U/\partial \eta^2)(\sin\hat{\phi})^2}{(1-M^2)\psi'^2}\frac{d\eta}{dn}. \tag{77}$$

Once again, we see that in elliptic domains ϑ is a monotonic function of s. However, we cannot exclude the possibility that level lines of η are closed, since ϑ need not be single-valued.

To summarize, we have demonstrated that the most general MHD equilibrium problem (with parallel $\mathbf{B}_p$ and $\mathbf{v}_p$) can be reduced to scalar equations of mixed type in natural coordinates, and all the specific details are incorporated in the form of the profile functions Π_i that determine the coefficient $U(\xi,\eta)$. Now we are ready to analyse the possibilities of finding exact solutions of the intrinsic equilibrium equations.

5. Equilibria with constant profiles Π_i

As a starting point of our analysis of the intrinsic equilibrium equations, we consider the case when all three profiles Π_i, $i = 1,2,3$, are constant. Without any loss of generality, we assume that $\Pi_1 = 1$, $\Pi_2 = A$ and $\Pi_3 = B$. Although even in this case the equilibrium equations remain highly non-trivial, some of their solutions can be found explicitly since in the case in question the profile function $U(\xi,\eta)$ is separable, $U(\xi,\eta) = e(\xi)f(\eta)$. The corresponding explicit solutions describe the MHD counterparts of several celebrated flows, such as the Prandtl–Meyer and Ringleb flows.

5.1. The Prandtl–Meyer flows

The development of Sec. 3.1 makes sense provided that $\partial\vartheta/\partial\xi \neq 0$, which is a tacit assumption there. In this subsection we study the opposite possibility and assume that $\partial\vartheta/\partial\xi = 0$ identically. We show that in the case in question we recover the classical Prandtl–Meyer solutions (Sedov 1965), as well as some more general solutions of a similar kind. These solutions can exist only in the hyperbolic domain. An alternative treatment of the MHD Prandtl–Meyer flows can be found in Kulikovskiy and Lyubimov (1965).

When $\partial\vartheta/\partial\xi = 0$, or equivalently $\vartheta = \vartheta(\eta)$, the following condition has to be satisfied in order to avoid a contradiction in (39):

$$\frac{\partial\vartheta}{\partial\eta} = \pm\left(\frac{1}{U}\frac{\partial^2 U}{\partial\eta^2}\right)^{1/2}. \tag{78}$$

This condition can be satisfied under very restrictive circumstances, namely when $U(\xi,\eta)$ is separable, $U(\xi,\eta) = e(\xi)f(\eta)$, or, more generally, when $U(\xi,\eta) = e_1(\xi)f_1(\eta) + e_2(\xi)f_2(\eta)$, where $f_1''/f_1 = f_2''/f_2$.

When the profiles Π_i, $i = 1,2,3$, are constant and ψ is used as a label (i.e. $\psi' = 1$), (56) and (58) show that U is a function of η only, $U = U(\eta)$ (albeit possibly a multivalued one). We assume below that this is the case.

For a Prandtl–Meyer flow, (39) and (40) show that

$$\frac{\partial\zeta}{\partial\xi} = \exp(i\vartheta)\left[-i\frac{dU}{d\eta} \pm U\left(\frac{1}{U}\frac{d^2U}{d\eta^2}\right)^{1/2}\right], \tag{79}$$

$$\frac{\partial\zeta}{\partial\eta} = \exp(i\vartheta)\,\Phi(\xi,\eta), \tag{80}$$

where Φ is a *real-valued* function of its arguments. The compatibility condition for (79) and (80) yields the following expression for Φ:

$$\begin{aligned}\Phi &= \int\left(U\frac{d^2\vartheta}{d\eta^2} + 2\frac{dU}{d\eta}\frac{d\vartheta}{d\eta}\right)d\xi + \omega \\ &= -i\int\left\{\exp(-i\vartheta)\frac{d^2[\exp(i\vartheta)U]}{d\eta^2}\right\}d\xi + \omega,\end{aligned} \tag{81}$$

where $\omega = \omega(\eta)$ is a free *real-valued* function of η.

Integration of (80) yields

$$\zeta - \zeta_0 = -i\int\int\frac{d[\exp(i\vartheta)\,U]}{d\eta}\,d\xi + \int\exp(i\vartheta)\,\omega\,d\eta, \tag{82}$$

where ζ_0 is an arbitrary constant.

Equation (71) shows that for Prandtl–Meyer flows the curves η = const are characteristics of the first family. Moreover, by virtue of (82) these characteristics are straight lines in the physical plane. Unfortunately, (71) does not determine the characteristics, of the second family. To find these characteristics we have to use the fact that streamlines bisect the angle between characteristics of different families. A relatively straightforward computation shows that in the (ξ,η) plane these characteristics are given by the equation

$$\xi^4 = \frac{\Upsilon}{U^3\,d^2U/d\eta^2}, \tag{83}$$

where Υ is an arbitrary positive constant.

It is clear that when $\omega(\eta) = 0$ all the characteristics of the first family emanate from one point $\zeta = \zeta_0$, and the flow represents an expansion fan connecting two uniform flows.

In Fig. 3(a) we show the super-Alfvénic flow with $(A,B) = (0.7, 0.001)$ in the strip with coordinates $(\log\xi, X)$, where $X = M^{-2}$. In this strip the streamlines are the horizontal lines of the form $\log\xi = \delta_{h,j}$, and the characteristics of the first family are the vertical lines of the form $M^{-2} = \delta_{v,i}$. We choose the horizontal lines to be equidistant,

$$\delta_{h,j+1} - \delta_{h,j} = \text{const}, \tag{84}$$

and the vertical line to be such that

$$U^3[\eta(\delta_{v,i+1}^{-1})]\frac{d^2U[\eta(\delta_{v,i+1}^{-1})]}{d\eta^2} - U^3[\eta(\delta_{v,i}^{-1})]\frac{d^2U[\eta(\delta_{v,i}^{-1})]}{d\eta^2} = \text{const}. \tag{85}$$

Such a choice guarantees that the characteristics of the second family intersect consecutive vertices of the grid formed by the streamlines and the characteristics of the first family. The image of the super-Alfvénic strip in the physical plane is shown in Fig. 3(b). The corresponding fan connects the sonic flow parallel to the line

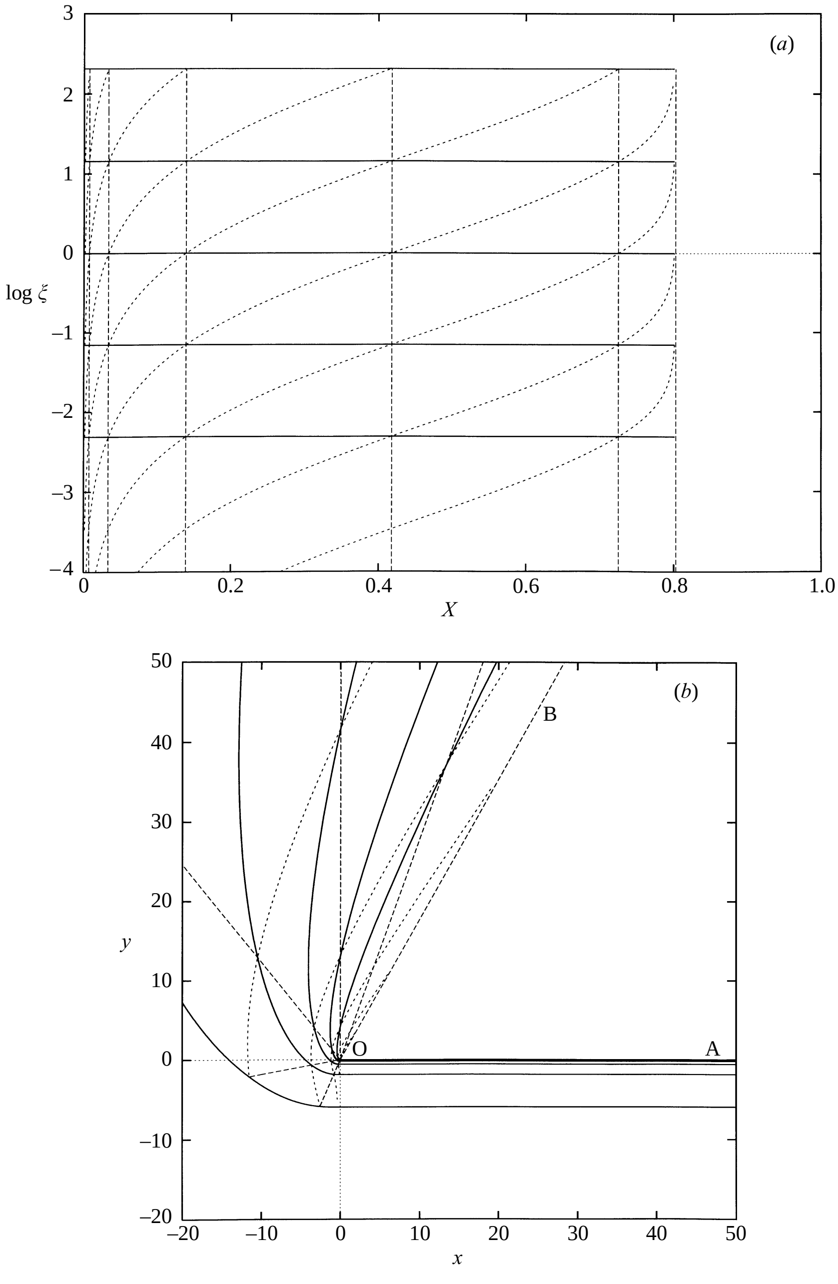
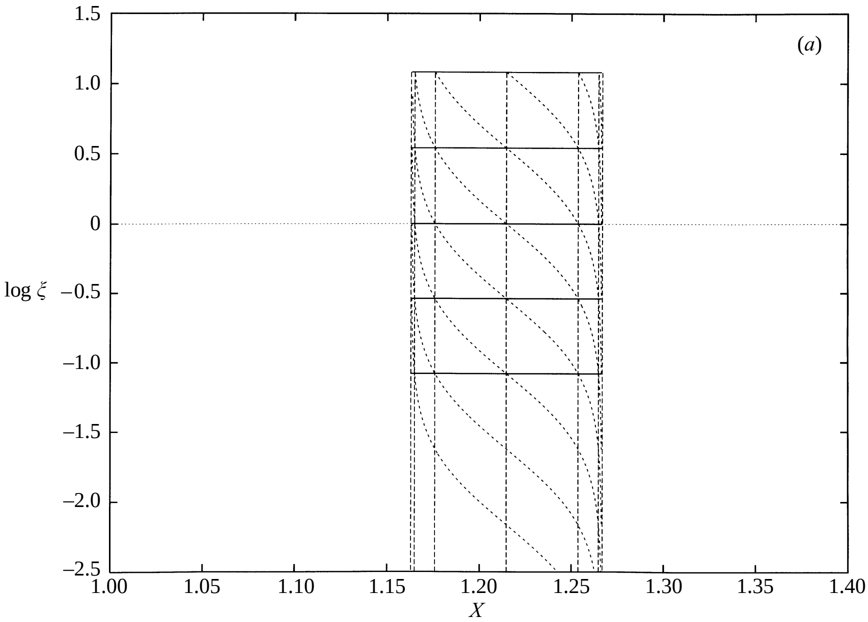


Figure 3. The largest super-Alfvénic Prandtl–Meyer fan with $(A, B) = (0.7, 0.001)$, (a) in the $(X, \log\xi)$ plane and (b) in the physical (x, y) plane: ——, streamlines; – – –, first family; - - - -, second family.

OA and the infinitely fast flow asymptotically parallel to the line OB. The interior of the angle AOB is inaccessible to the flow. Any other fan with $(A, B) = (0.7, 0.001)$ can be obtained from the fan in question by cutting a part of it bounded by two characteristics of the first family.



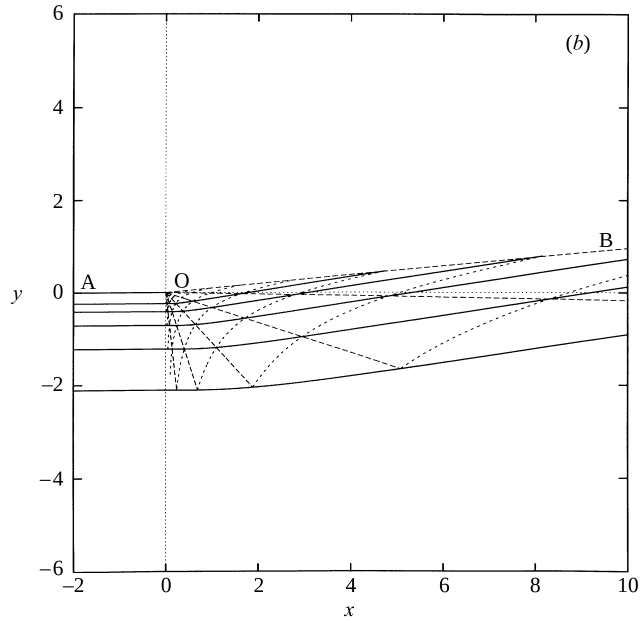


Figure 4. The largest sub-Alfvénic Prandtl–Meyer fan with $(A, B) = (0.7, 0.001)$, (a) in the $(X, \log \xi)$ plane and (b) in the physical (x, y) plane: ——, streamlines; – – –, first family; - - - -, second family.

In Figs $4(a, b)$ we show the sub-Alfvénic Prandtl–Meyer fan with $(A, B) = (0.7, 0.001)$. Figure $4(a)$ is similar to its super-Alfvénic counterpart. The fan in Fig. $4(b)$ connects two sonic flows parallel to the lines OA and OB respectively. An interesting feature of the sub-Alfvénic case is the fact that the characteristics are

(a)

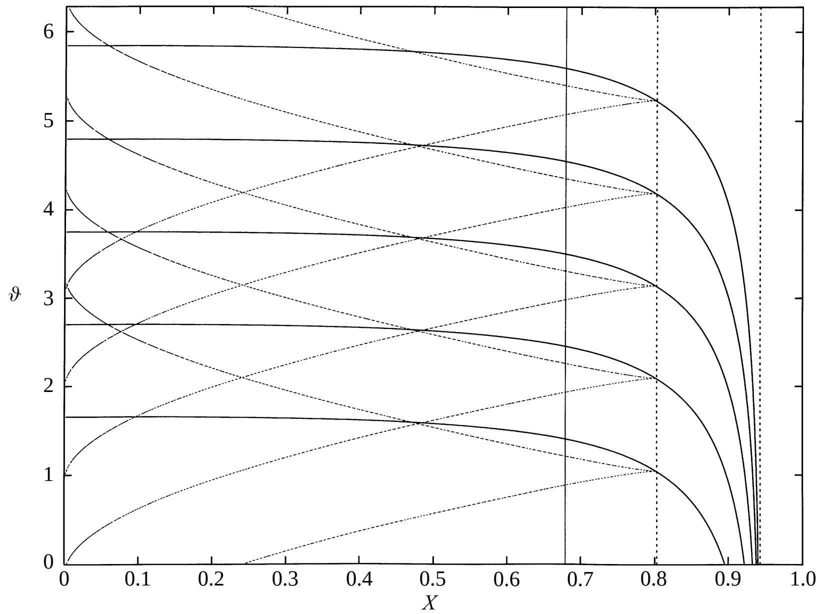


Figure 5 (a). For legend see facing page.

asymptotically parallel to the streamlines in the vicinity of the transition line OB. As before, the interior of the angle AOB is inaccessible to the flow.

When $\omega(\eta) \neq 0$ the characteristics of the first family are tangents to a certain envelope curve, and the fan describes a flow around this curve.

5.2. The Ringleb flows

When U is independent of ξ, $U = f(\eta)$ (which happens when Π_i = constant, $i = 1, 2, 3$, and ψ is used as a label of flux and streamsurfaces), (46) becomes linear and assumes the form

$$U\frac{\partial^2\xi}{\partial\eta^2} - \frac{d^2U}{d\eta^2}\frac{\partial^2\xi}{\partial\vartheta^2} + 2\frac{dU}{d\eta}\frac{\partial\xi}{\partial\eta} = 0. \tag{86}$$

This equation can be considered as the generalized Chaplygin equation. It turns out that the hodograph equations can also be linearized when the coefficient function U is a separable function of its arguments, i.e. $U(\xi, \eta) = e(\xi)f(\eta)$. We achieve this linearization by replacing the original dependent variable ξ by a new one, $\tilde{\xi} = \int e(\xi)\, d\xi$. Converting partial derivatives of ξ with respect to η and ϑ into partial derivatives of $\tilde{\xi}$ with respect to η and ϑ, we can rewrite (46) in the form (86) with ξ replaced by $\tilde{\xi}$; the latter equation is obviously linear. For (86) the characteristics are independent of ξ. They are given by the equations

$$\vartheta = \pm\int\left(\frac{1}{U}\frac{d^2U}{d\eta^2}\right)^{1/2} d\eta. \tag{87}$$

The generalized Chaplygin equation (86) possesses a large variety of special solutions. To give some examples, we consider two special solutions, which repre-

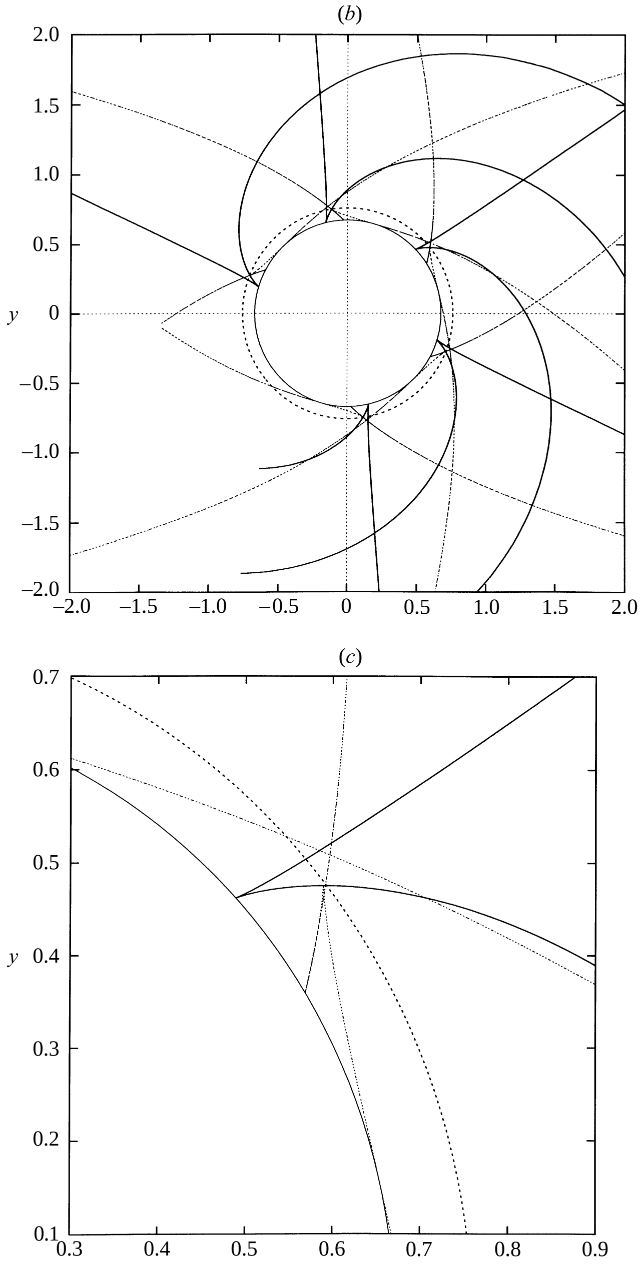


Figure 5. The super-Alfvénic spiral flow with $(A, B) = (0.7, 0.001)$ and $(\alpha, \beta) = (1.0, 0.1)$, ($a$) in the hodograph (X, ϑ) plane and (b) in the physical (x, y) plane: - - - -, transition; ——, limit; **——**, streamline; ······, characteristic. (c) A detail of (b) showing streamlines and characteristics in the vicinity of the limit line.

sent spiral plasma flows and plasma flows around a semi-infinite plate. For two-dimensional irrotational fluid flows the corresponding solutions were found by Ringleb (1941) (for an alternative discussion of spiral plasma flows see Seebass 1961; Webb *et al.* 1994).

(a)

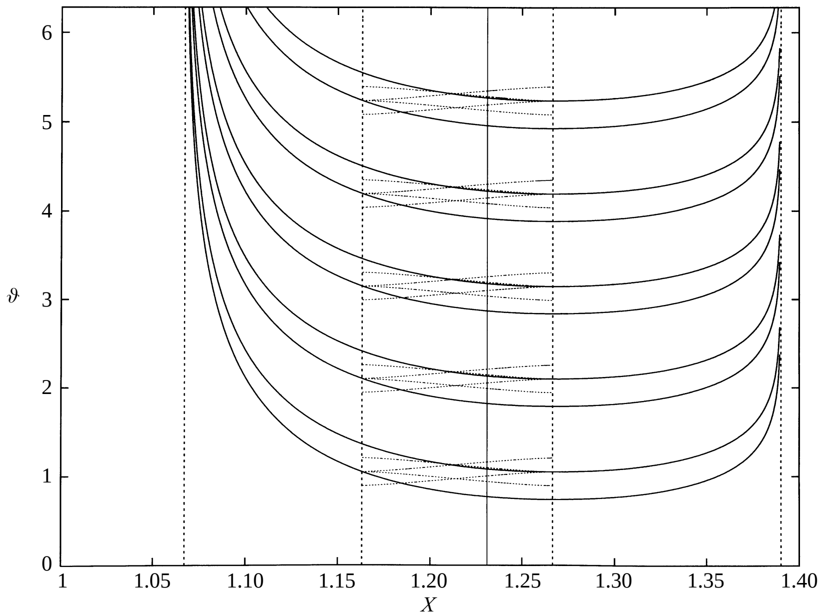


Figure 6 (*a*). For legend see facing page.

5.2.1. Spiral flows. First, we describe spiral flows. A straightforward substitution shows that (86) has the solution

$$\xi(\eta, \vartheta) = \alpha\vartheta + \beta \int \frac{d\eta}{U^2(\eta)}, \tag{88}$$

where α and β are arbitrary constants. In the (η, ϑ) plane the streamlines are given by the condition

$$\alpha\vartheta = \xi - \beta \int \frac{d\eta}{U^2(\eta)}, \qquad \text{where} \quad \xi = \text{const}, \tag{89}$$

the sonic lines are straight lines determined by the conditions $(d^2U/d\eta^2)/U = 0, \pm\infty$, while the characteristics are given by (87). Substitution of (88) into (49) yields the following equation for the location of the limit line:

$$\alpha^2 U^3(\eta) \frac{d^2U(\eta)}{d\eta^2} = \beta^2. \tag{90}$$

Finally, integration of (47) allows one to express (x, y) in terms of (η, ϑ) as follows:

$$x = -\alpha \frac{dU}{d\eta} \cos\vartheta - \frac{\beta}{U} \sin\vartheta, \qquad y = \frac{\beta}{U} \cos\vartheta - \alpha \frac{dU}{d\eta} \sin\vartheta. \tag{91}$$

Using these equations, we can find the flow in the physical plane. For $(A, B) = (0.7, 0.001)$ the flow structure in the hodograph and physical planes is given in Figs 5 and 6. Figures 5(*a*) and 6(*a*) show the streamlines, sonic lines, limit lines and characteristics in the hodograph plane. As mentioned earlier, the flow exists in two disjoint strips: the super-Alfvénic strip $M_3^2 \leqslant M^2, -\infty < \vartheta < \infty$, and the sub-Alfvénic strip $M_1^2 \leqslant M^2 \leqslant M_2^2, -\infty < \vartheta < \infty$. In both strips the streamlines, characteristics, etc. are well behaved. Each strip is mapped onto the physical plane

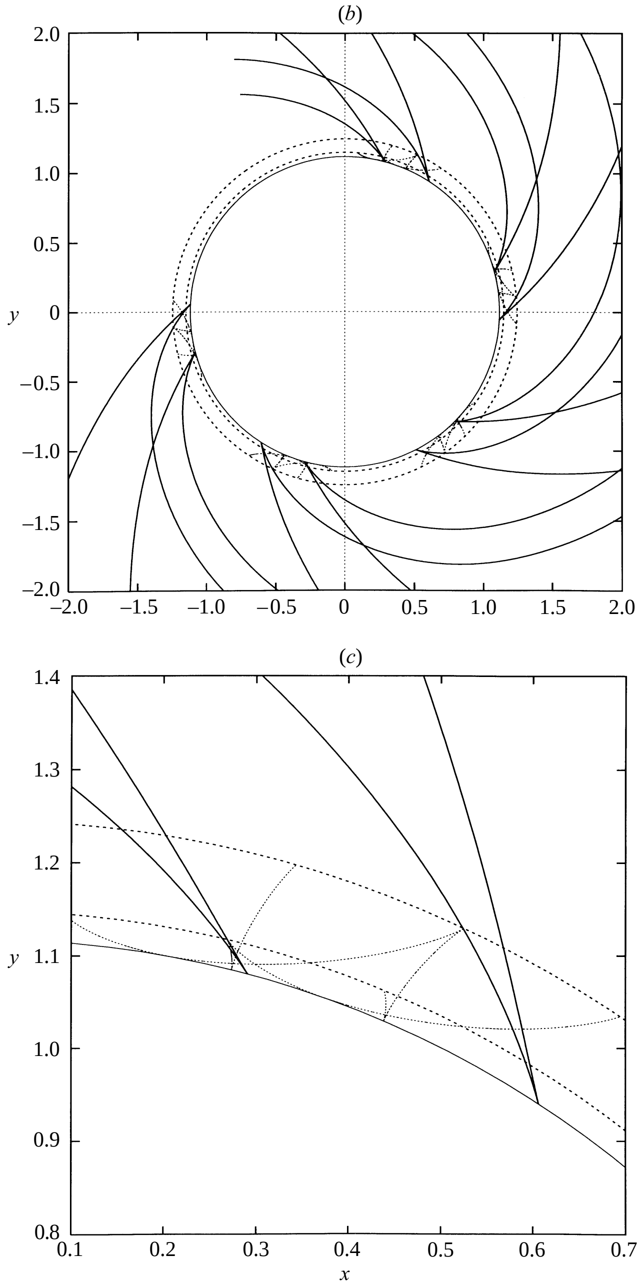


Figure 6. The sub-Alfvénic spiral flow with $(A, B) = (0.7, 0.001)$ and $(\alpha, \beta) = (1.0, 0.1)$, ($a$) in the hodograph (X, ϑ) plane and (b) in the physical (x, y) plane: - - - -, transition; ——, limit; ▬▬, streamline; ·····, characteristic. (c) A detail of (b) showing streamlines and characteristics in the vicinity of the limit line.

separately (see Figs 5b, 6b). Under the mapping (91), the images of the sonic and limit lines are circles. Parts of the same strip separated by the limit line are mapped onto the same part of the physical plane. Thus, for each strip, we deal with two flows that coexist outside the circle being the image of the limit line. Neither of

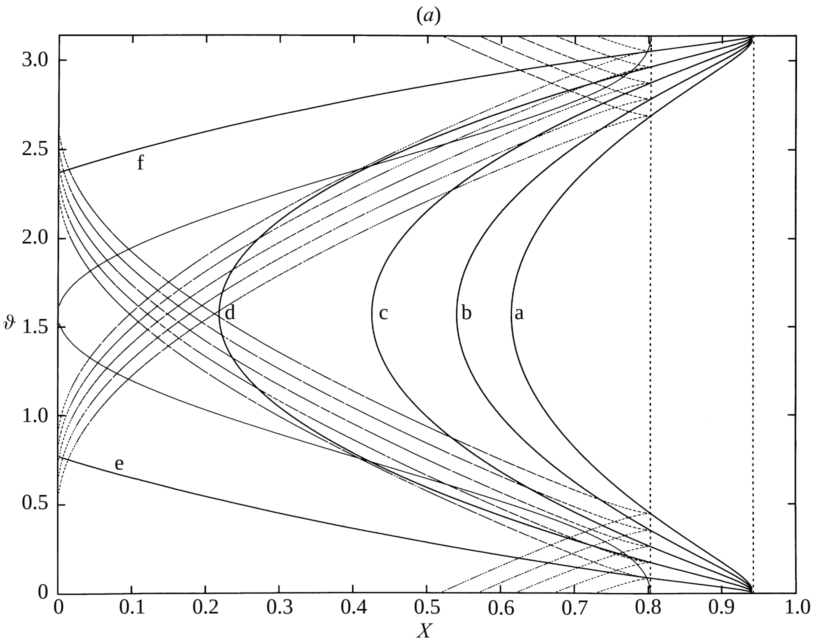


Figure 7 (*a*). For legend see facing page.

these flows can penetrate within that circle. Inspection of Figs 5(*b*) and 6(*b*) shows that the limit circle is the envelope for one family of characteristics and a barrier for the other family. For the sub-Alfvénic strip, both flows are transonic, and change type from elliptic to hyperbolic on the corresponding circular transition lines. For the super-Alfvénic strip, one flow is transonic and changes its type on the circular transition line, while the other is purely hyperbolic.

5.2.2. Flows around a plate. Next, we consider the Ringleb flows around a plate. In the hodograph plane these flows corresponds to particular solutions of (86) of the form

$$\xi(\eta,\vartheta) = \frac{\sin\vartheta}{U(\eta)}. \tag{92}$$

The corresponding streamlines are given by the condition

$$\sin\vartheta = \xi U(\eta), \qquad \text{where} \quad \xi = \text{const}, \tag{93}$$

while the sonic lines and characteristics have the same form as before. The limit line is determined by the equation

$$\cos(2\vartheta) = \frac{(dU/d\eta)^2 - U\,d^2U/d\eta^2}{(dU/d\eta)^2 + U\,d^2U/d\eta^2}. \tag{94}$$

The mapping $(\eta,\vartheta) \to (x,y)$ has the form

$$x = -\int \frac{(dU/d\eta)^2}{U^2}\,d\eta - \frac{dU/d\eta}{2U}[\cos(2\vartheta)+1], \qquad y = -\frac{dU/d\eta}{2U}\sin(2\vartheta). \tag{95}$$

The Ringleb flows with $(A,B) = (0.7, 0.001)$ in both hodograph and physical planes are given in Figs 7 and 8. In contrast to the previous case, the admissible part of the hodograph plane now consists of two rectangles $M_3^2 \leqslant M^2, 0 < \vartheta < \pi$ and

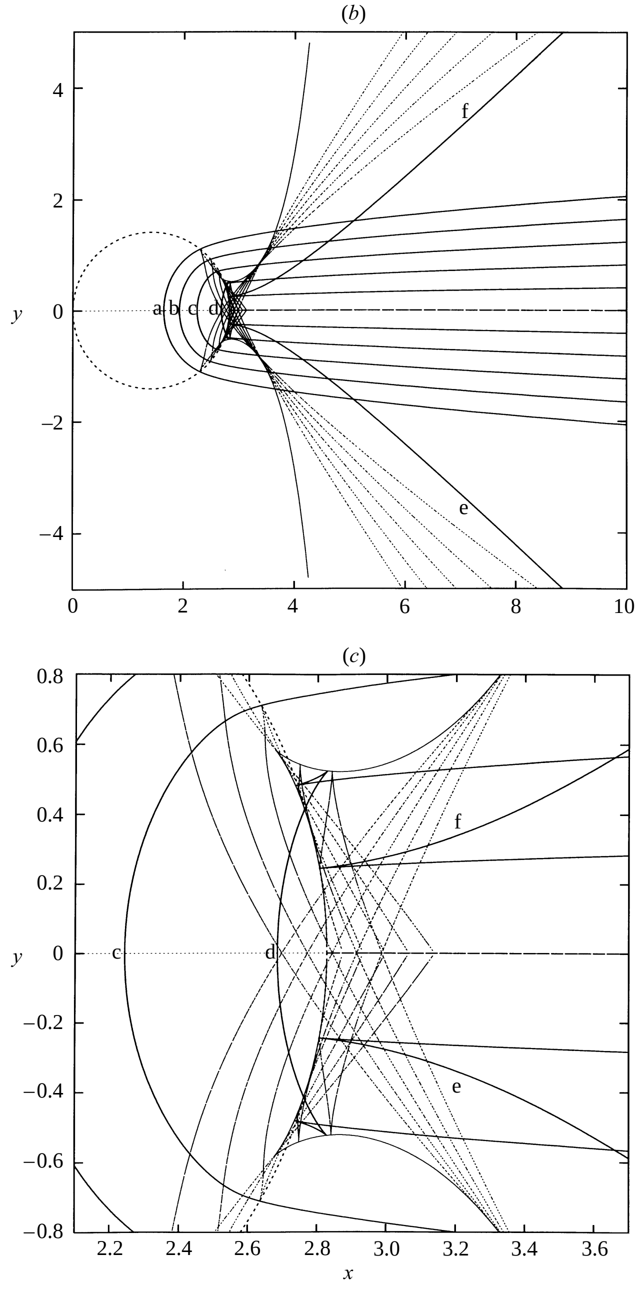


Figure 7. The super-Alfvénic flow around a plate with $(A, B) = (0.7, 0.001)$, (a) in the hodograph (X, ϑ) plane and (b) in the physical (x, y) plane: - - - - - -, transition; ——, limit; ━━, streamline; ······, characteristic; — —, plate. (c) A detail of (b). Streamlines 'a','b','c' are physically admissible; streamlines 'd','e','f' are not.

$M_1^2 \leqslant M^2 \leqslant M_2^2, 0 < \vartheta < \pi$, since the flows around a plate rotate by no more then π. The plate itself corresponds to the top and bottom sides of these rectangles. As before, the streamlines, characteristics, etc. are smooth in the hodograph plane (see Figs 7*a*, 8*a*).

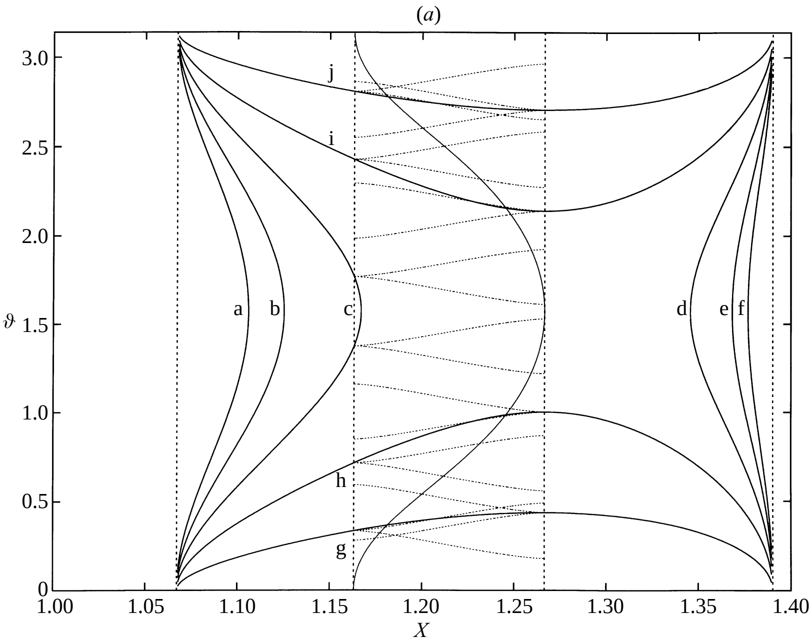


Figure 8 (a). For legend see facing page.

First, we consider the mapping of the super-Alfvénic rectangle onto the physical plane, since the resulting flows closely resemble their fluid counterparts, and we can build upon the conventional intuition (see Figs 7b, c). The image of the plate is part of the positive horizontal semi-axis, while the image of the sonic line is a circle passing through the origin. The image of the limit line is a curve passing through the tip of the plate and extending to infinity; it has two cusps. The physical plane with a branch cut along the plate supports several overlapping flows. The first flow is of mixed type. Its streamlines are of two different kinds:

(i) the streamlines not crossing the central branch of the limit line, such as 'a','b' and 'c';

(ii) the streamlines ending on the central branch, such as 'd','e' and 'f'.

The streamlines of the first kind start and end at infinity; they do not feel the presence of the limit line, and represent a physically admissible flow around the plate in question that is partly elliptic and partly hyperbolic. This flow, however, does not cover the entire plane. It exists only in the exterior of the domain bounded by the streamline passing through the cusp points of the limit line. Inside this domain several flows coexist, namely the first flow of mixed type with the streamlines starting (or ending) at infinity and ending (or starting) at the limit line, the second flow of hyperbolic type with similar behaviour of the streamlines, and, finally, the third flow with the streamlines both starting and ending at the limit line. Inspection of Figs 7(b, c) shows that the limit line is the envelope for one family of characteristics, and the barrier for the other family. The image of a smooth streamline in the hodograph plane can have up to four cusps in the physical plane, like the curve 'd'.

The mapping of the sub-Alfvénic rectangle has some interesting features (see

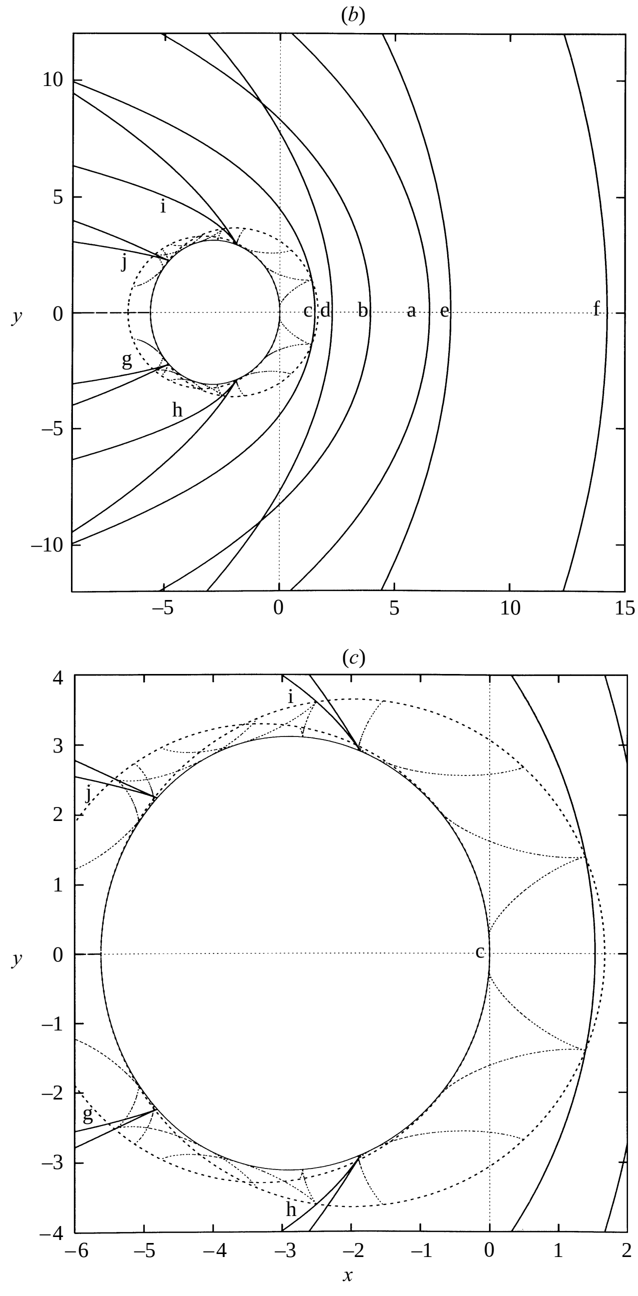


Figure 8. The sub-Alfvénic flow around a plate with $(A, B) = (0.7, 0.001)$, (a) in the hodograph (X, ϑ) plane and (b) in the physical (x, y) plane: - - - - - -, transition; ——, limit; **——**, streamline; ⋯⋯, characteristic; — —, plate. (c) A detail of (b). Streamlines 'a','b','c','d','e','f' are physically admissible; streamlines 'g','h','i','j' are not.

Figs 8b, c). The image of the plate is now part of the negative semi-axis, the images of the sonic lines are two circles, while the image of the limit line is a closed curve touching both sonic lines. In the physical plane two flows coexist. Both are of mixed type. As before, the limit line is the envelope for the characteristics of an appropriate family. Streamlines not crossing the limit line in the hodograph plane, such as ‘a’,‘b’,‘c’,‘d’,‘e’ and ‘f’ are physically admissible. Streamlines that do not cross the limit line, such as ‘h’,‘g’,‘i’ and ‘j’, are smooth in the hodograph plane but have cusps in the physical plane and are not physically admissible. Streamlines ‘a’,‘b’,‘c’ represent one type of flow, which can be both elliptic and hyperbolic, while streamlines ‘d’,‘e’,‘f’ represent another type of flow, which can only be elliptic.

6. Symmetries of the equilibrium equations

In this section we discuss (42) from the group-theoretical viewpoint. First of all, we notice that if ϑ is a solution of (42) then $\tilde{\vartheta} = \pm\vartheta + \vartheta_0$, where ϑ_0 is a constant, is a solution as well. This statement is due to the fact that ϑ is the angle between $\nabla\xi$ and the x axis and that reflection of the equilibrium in this axis, or its rotation by a certain angle, produces a new equilibrium.

To advance further, we assume that the coefficient function U is scale-invariant. A simple analysis of (56) and (58) shows that this is possible only when the profiles Π_i, $i = 1, 2, 3$, have the same shape, given by a certain master profile π (this restriction is not applicable in the limiting cases when the coefficient function U is given either by (61) or by (64)). Without loss of generality, we assume that $\Pi_1 = \pi$, $\Pi_2 = A\pi$ and $\Pi_3 = B\pi$, where A and B are certain positive constants. We have the following expressions for η and U:

$$\eta = \pi(\xi)\left(\frac{1}{M^4} - \frac{A}{M^{2(\gamma+1)}} + \frac{\gamma-1}{\gamma}\frac{A}{M^{2\gamma}}\right), \tag{96}$$

$$U = (1-M^2)\left\{2\psi'^2(\xi)\pi(\xi)\left[\frac{1}{M^4} - \frac{A}{M^{2(\gamma+1)}} - \frac{B}{(1-M^2)^2}\right]\right\}^{1/2}. \tag{97}$$

Equation (96) shows that M^2 can be thought of as a function of $\tau = \eta/\pi(\xi)$. Accordingly, U can be written in the form

$$U(\xi,\eta) = \sigma(\xi)u\left(\frac{\eta}{\pi(\xi)}\right) = \sigma(\xi)u(\tau), \tag{98}$$

where

$$\sigma(\xi) = \psi'(\xi)[\pi(\xi)]^{1/2}. \tag{99}$$

It is clear that in the limiting cases the coefficient function can be written in the form (98) as well. When the magnetic flux function ψ rather than a generic ξ is used as a label, $\sigma = \pi^{1/2}$, and the situation becomes somewhat simpler. However, we prefer to use ξ to be able to cover the classical limiting cases of static plasma equilibria and compressible fluid flows.

We need to restrict the choice of the profile functions π and σ in such a way that (42) has scale-invariant solutions of the form

$$\vartheta(\xi,\eta) = \theta(\tau) - \kappa\log[\pi(\xi)]. \tag{100}$$

First, we introduce coordinates (ξ,τ) instead of (ξ,η), and write

$$\left.\frac{\partial}{\partial\xi}\right|_\eta = \left.\frac{\partial}{\partial\xi}\right|_\tau - \frac{\pi'}{\pi}\tau\left.\frac{\partial}{\partial\tau}\right|_\xi, \qquad \left.\frac{\partial}{\partial\eta}\right|_\xi = \frac{1}{\pi}\left.\frac{\partial}{\partial\tau}\right|_\xi, \tag{101}$$

$$\left.\frac{\partial U}{\partial \eta}\right|_{\xi} = \frac{\sigma}{\pi}u', \qquad \left.\frac{\partial^2 U}{\partial \eta^2}\right|_{\xi} = \frac{\sigma}{\pi^2}u'', \tag{102}$$

$$\left.\frac{\partial \vartheta}{\partial \xi}\right|_{\eta} = -\frac{\pi'}{\pi}(\tau\theta' + \kappa), \qquad \left.\frac{\partial \vartheta}{\partial \eta}\right|_{\chi} = \frac{1}{\pi}\theta'. \tag{103}$$

Using these equations, we can write the corresponding equation for θ as

$$-\left(\frac{\sigma}{\pi\pi'}\right)'\frac{u'' - u\theta'^2}{\tau\theta' + \kappa} + \frac{\sigma}{\pi^2}\tau\left(\frac{u'' - u\theta'^2}{\tau\theta' + \kappa}\right)' + \frac{\sigma}{\pi^2}\frac{1}{u}(u^2\theta')' = 0. \tag{104}$$

It is clear that (104) is consistent if and only if

$$\left(\frac{\sigma}{\pi\pi'}\right)' = \mu\frac{\sigma}{\pi^2}, \tag{105}$$

where μ is a certain constant.

When the condition (105) is satisfied, the ξ-dependent factors can be cancelled out, and (104) can be reduced to an ODE for θ of the form

$$-\mu\frac{u'' - u\theta'^2}{\tau\theta' + \kappa} + \tau\left(\frac{u'' - u\theta'^2}{\tau\theta' + \kappa}\right)' + \frac{1}{u}(u^2\theta')' = 0. \tag{106}$$

Thus, in the case where the coefficient U in (42) is scale-invariant, we can find a family of scale-invariant solutions of this equation via integration of the corresponding ODE.

It is clear that for scale-invariant coefficient functions u the transition condition (67) can be written in the form

$$uu'' \begin{cases} < 0 & \text{(elliptic)}, \\ = 0 & \text{(parabolic)}, \\ > 0 & \text{(hyperbolic)} \end{cases} \tag{107}$$

so that transitions occur at fixed values of τ that are independent of ξ.

It is easy to verify that the condition (105) implies that

$$(\log\sigma)' = [\log(\pi^{\mu+1}\pi')]'. \tag{108}$$

Integration of this relation yields

$$\int \sigma(\tilde{\xi})\, d\tilde{\xi} \sim \begin{cases} \dfrac{[\pi(\xi)]^{\mu+2}}{\mu+2} & (\mu \neq -2), \\ \log[\pi(\xi)] & (\mu = -2). \end{cases} \tag{109}$$

It is clear that power functions always satisfy (109). Besides, exponential functions satisfy this relation as well. Apart from certain exceptional cases, the coefficient functions U of the form (98) with π and σ related via (109) are the most general profiles allowing integration of (42) via group-theoretical methods.

If ψ is used as a label then $\sigma = \pi^{1/2}$, and (109) yields

$$\left.\begin{aligned} &\pi \sim |\psi|^{2/(3+2\mu)}, \quad \sigma \sim |\psi|^{1/(3+2\mu)}, \quad \text{if } \mu \neq -\tfrac{3}{2}, \\ &\log\pi \sim |\psi|, \quad \log\sigma \sim |\psi| \quad \text{if } \mu = -\tfrac{3}{2}. \end{aligned}\right\} \tag{110}$$

This is a restriction on the form of $\pi(\psi)$, rather than a relation between $\pi(\xi)$ and $\sigma(\xi)$. The relations (110) show that the profile π should be either a power or an exponential function for (42) written in terms of ψ and η to have self-similar solutions.

7. Self-similar solutions

In this section we study general self-similar *plasma* flows governed by (106). An alternative treatment of self-similar *fluid* flows can be found in Sedov (1965). We consider power profiles π and σ of the form

$$\pi(\xi) = |\xi|^{2/(3+2\mu)}, \qquad \sigma(\xi) = |\xi|^{1/(3+2\mu)}, \qquad \mu \neq -\tfrac{3}{2}. \tag{111}$$

Exponential profiles can be considered as a special case and treated along similar lines. Introducing the auxiliary variable

$$Y = \frac{1}{(\tau u\theta' + \kappa u)^2}, \tag{112}$$

we rewrite (106) as

$$Y' + 2\left(\log\left|\frac{\tau^2 uu'' - \kappa^2 u^2}{\tau^{2+\mu}}\right|\right)' Y - \frac{2(3+2\mu)\kappa u}{\tau(\tau^2 uu'' - \kappa^2 u^2)} Y^{1/2} + \frac{2[\tau u' + (1+\mu)u]}{\tau u(\tau^2 uu'' - \kappa^2 u^2)} = 0. \tag{113}$$

Thus, when the coefficient U in (42) is scale-invariant, we can find scale-invariant solutions of this equation via numerical integration. Since (113) is an inhomogeneous linear equation for Y provided that $\kappa = 0$, particular families of self-similar solutions can be found explicitly up to two quadratures (see below).

Needless to say, self-similar solutions are written in natural coordinates ξ and η (or, more precisely, ξ and τ). In order to represent these solutions in the original variables x and y we have to find ζ from (41). Writing $d\eta$ in the form

$$d\eta = |\xi|^{2/(3+2\mu)}\tau\left(\frac{2}{3+2\mu}\frac{d\xi}{\xi} + \frac{d\tau}{\tau}\right), \tag{114}$$

i.e. using ξ and τ as independent variables, and applying (101)–(103), we rewrite this equation as

$$d\zeta = |\xi|^{(2+2\mu-2i\kappa)/(3+2\mu)}\tau\exp(i\theta) \times \frac{[-iu'(\theta' + \kappa/\tau) + \kappa u\theta'/\tau + u'']d\xi/\xi + (\tfrac{3}{2}+\mu)(u'' - u\theta'^2)d\tau/\tau}{\tau\theta' + \kappa}. \tag{115}$$

Integration of this equation yields

$$\zeta - \zeta_0 = \frac{3+2\mu}{2+2\mu-2i\kappa}|\xi|^{(2+2\mu-2i\kappa)/(3+2\mu)}\exp(i\theta)\,\frac{-iu'(\tau\theta'+\kappa) + \kappa u\theta' + \tau u''}{\tau\theta' + \kappa}, \tag{116}$$

where ζ_0 is an arbitrary constant. Equation (116) provides the relation between natural coordinates (ξ, τ) and Cartesian coordinates (x, y) that we sought. Introducing polar coordinates (r, ϕ) in the ζ plane centred at ζ_0, we write $\zeta - \zeta_0 = r\exp(i\phi)$. Substituting this expression into (116), we obtain the following expressions for r and ϕ:

$$r = |\xi|^{(2+2\mu)/(3+2\mu)}R(\tau), \qquad \phi = \Phi(\tau) - \frac{2\kappa}{3+2\mu}\log|\xi|, \tag{117}$$

where

$$\left.\begin{aligned} R(\tau) &= \frac{|3+2\mu|}{[(2+2\mu)^2+4\kappa^2]^{1/2}}\,\frac{[u'^2(\tau\theta'+\kappa)^2+(\tau u''+\kappa u\theta')^2]^{1/2}}{|\tau\theta'+\kappa|},\\ \Phi(\tau) &= \theta-\arctan\!\left(\frac{\tau u'\theta'+\kappa u'}{\tau u'+\kappa u\theta'}\right)+\arctan\!\left(\frac{\kappa}{1+\mu}\right). \end{aligned}\right\} \qquad (118)$$

Equations (117) clearly imply that all the streamlines given by the condition $\xi =$ const are similar, so that the solution we found is indeed scale-invariant.

Equation (116) allows us to construct the streamlines of the flow by keeping ξ fixed and varying τ. It is of great interest to us to be able to construct the characteristics in the regions where (42) is hyperbolic. First, we find the characteristic curves in the (ξ, τ) plane and then use (116) to find the images of the corresponding curves in the ζ plane. We first rewrite (70) determining the characteristics in terms of ξ and τ:

$$\frac{\partial\vartheta}{\partial\xi}\,d\xi+|\xi|^{2/(3+2\mu)}\tau\left[\pm\left(\frac{1}{U}\frac{\partial^2 U}{\partial\eta^2}\right)^{1/2}+\frac{\partial\vartheta}{\partial\eta}\right]\left(\frac{2}{3+2\mu}\frac{d\xi}{\xi}+\frac{d\tau}{\tau}\right)=0. \qquad (119)$$

We then use (101)–(103) to rewrite this equation in the form

$$\frac{2}{3+2\mu}\left[\pm\left(\frac{u''}{u}\right)^{1/2}-\frac{\kappa}{\tau}\right]\frac{d\xi}{\xi}+\left[\pm\left(\frac{u''}{u}\right)^{1/2}+\theta'\right]\frac{d\tau}{\tau}=0. \qquad (120)$$

Finally, we integrate this equation to obtain

$$|\xi|^{-2/(3+2\mu)}=\Upsilon\tau\exp\!\left[\int\frac{\tau\theta'+\kappa}{\pm\tau(u''/u)^{1/2}-\kappa}\,\frac{d\tau}{\tau}\right], \qquad (121)$$

where Υ is a certain positive constant. Substituting this relation between ξ and τ into (116), we obtain the characteristic curves in the ζ plane:

$$\begin{aligned}\zeta-\zeta_0=\frac{3+2\mu}{2+2\mu-2i\kappa}&\left\{\Upsilon\tau\exp\!\left[\int\frac{\tau\theta'+\kappa}{\pm\tau(u''/u)^{1/2}-\kappa}\,\frac{d\tau}{\tau}\right]\right\}^{-(1+\mu-i\kappa)}\\ &\times\exp(i\theta)\frac{-iu'(\tau\theta'+\kappa)+\kappa u\theta'+\tau u''}{\tau\theta'+\kappa}. \end{aligned} \qquad (122)$$

The above expressions become particularly simple when $\kappa = 0$. In this case (113) is an inhomogeneous linear equation for Y. We solve this equation via the method of variation of constants. First, we note that solutions of the homogeneous equation can be written as

$$Y=C\frac{\tau^{2\mu}}{(uu'')^2}, \qquad (123)$$

where C is an arbitrary constant. Accordingly, solutions of the inhomogeneous equation can be written in the same form, but with the factor C given by the integral

$$C=-2\int[\tau u'+(1+\mu)u]u''\frac{d\tau}{\tau^{3+2\mu}}. \qquad (124)$$

Once this integral has been found, we can use (112) and represent θ as

$$\theta=\pm\int\frac{u''}{C^{1/2}}\frac{d\tau}{\tau^{1+\mu}}. \qquad (125)$$

(a)

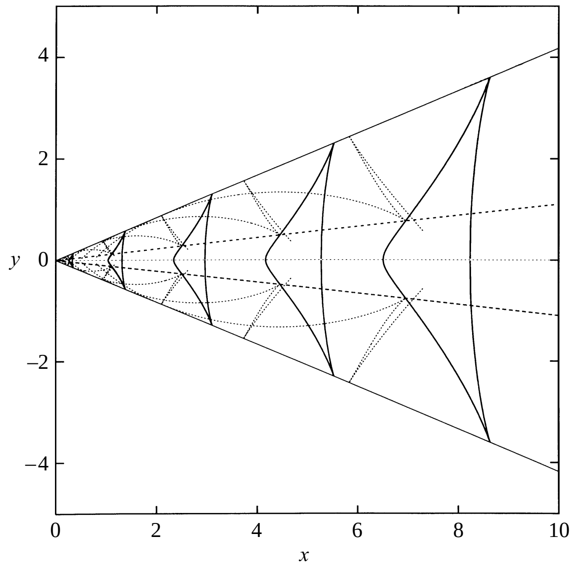


Figure 9 (*a*). For caption see page 96.

Substituting $\kappa = 0$ into (116)–(118) yields expressions for x, y, r and ϕ in terms of ξ and τ. In the hyperbolic domain, where $uu'' > 0$, the characteristics in the intrinsic and physical plane are given by (121) and (122) with $\kappa = 0$.

In general, it is difficult to find the integrals (124) and (125). However, this can be done explicitly for certain special values of μ. Suppose, for example, that $\mu = -2$. Introducing the notation $\omega = \tau u' - u$, we can represent C in the form

$$C = -\int (\omega^2)' d\tau = \Omega^2 - \omega^2, \tag{126}$$

where Ω is an arbitrary constant. Likewise, θ can be written as

$$\theta = \int \frac{\omega'}{(\Omega^2 - \omega^2)^{1/2}} d\tau = \arcsin\left(\frac{\omega}{\Omega}\right) + \theta_0, \tag{127}$$

where θ_0 is a constant; without loss of generality, we put $\theta_0 = 0$. The mapping $(\xi, \tau) \to (x, y)$ can be written as

$$\zeta - \zeta_0 = \frac{|\xi|^2}{2\Omega\tau}[\omega u + \Omega^2 - i(\Omega^2 - \omega^2)^{1/2} u]. \tag{128}$$

The equations for the characteristics in the intrinsic and physical planes assume the forms

$$|\xi|^2 = \Upsilon\tau \exp\left[\pm \int \left(\frac{uu''}{\Omega^2 - \omega^2}\right)^{1/2} d\tau\right], \tag{129}$$

$$\zeta - \zeta_0 = \frac{\Upsilon}{2\Omega} \exp\left[\pm \int \left(\frac{uu''}{\Omega^2 - \omega^2}\right)^{1/2} d\tau\right][\omega u + \Omega^2 - i(\Omega^2 - \omega^2)^{1/2} u] \tag{130}$$

respectively. Transitions from ellipticity to hyperbolicity occur when either $uu'' = 0$ or $uu'' = \infty$. It can be shown by virtue of (129) and (130) that in the first case the

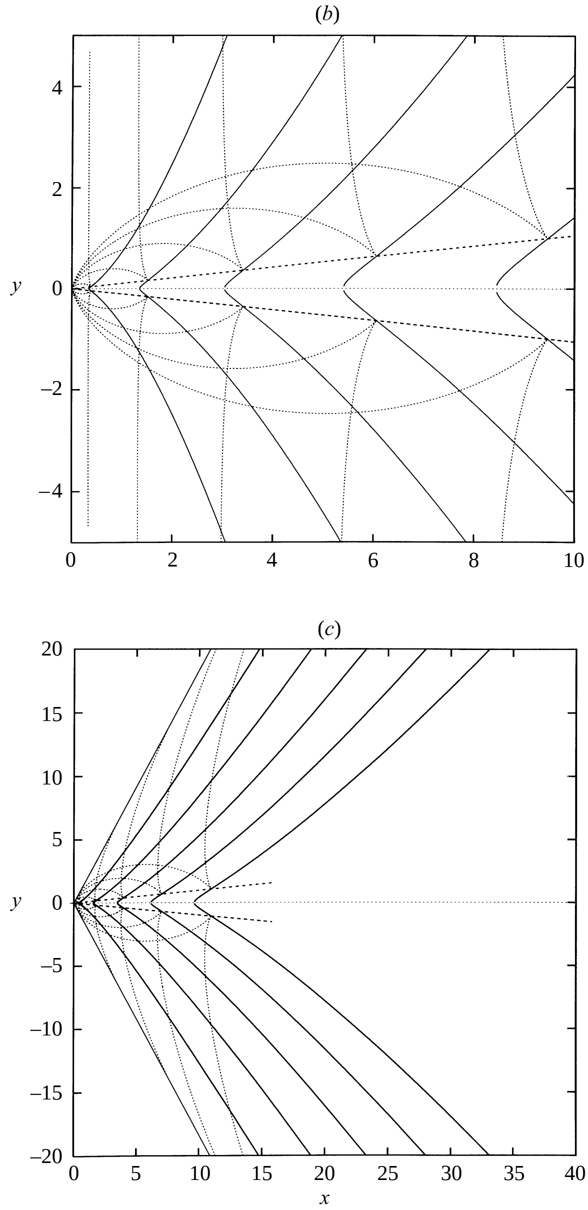


Figure 9 (b, c). For caption see page 96.

characteristics emanating from the transition line are orthogonal to the corresponding streamline, while in the second case they are parallel to this streamline. In Figs 9(a–e) we present plasma flows with $(A, B) = (0.7, 0.001)$ corresponding to several representative values of $\mathbf{\Omega}$. The flows cover the entire plane only for $\mathbf{\Omega} = \mathbf{\Omega}_{\text{cr}}$, where $\mathbf{\Omega}_{\text{cr}} = \omega(0)$. We emphasize that for $\mathbf{\Omega} = \mathbf{\Omega}_{\text{cr}}$ the streamlines asymptotically approach semicubic parabolas, while for $\mathbf{\Omega} \neq \mathbf{\Omega}_{\text{cr}}$ they approach straight lines. In principle, flows with $\mathbf{\Omega} = \mathbf{\Omega}_{\text{cr}}$ can be used as a model for galactic jets.

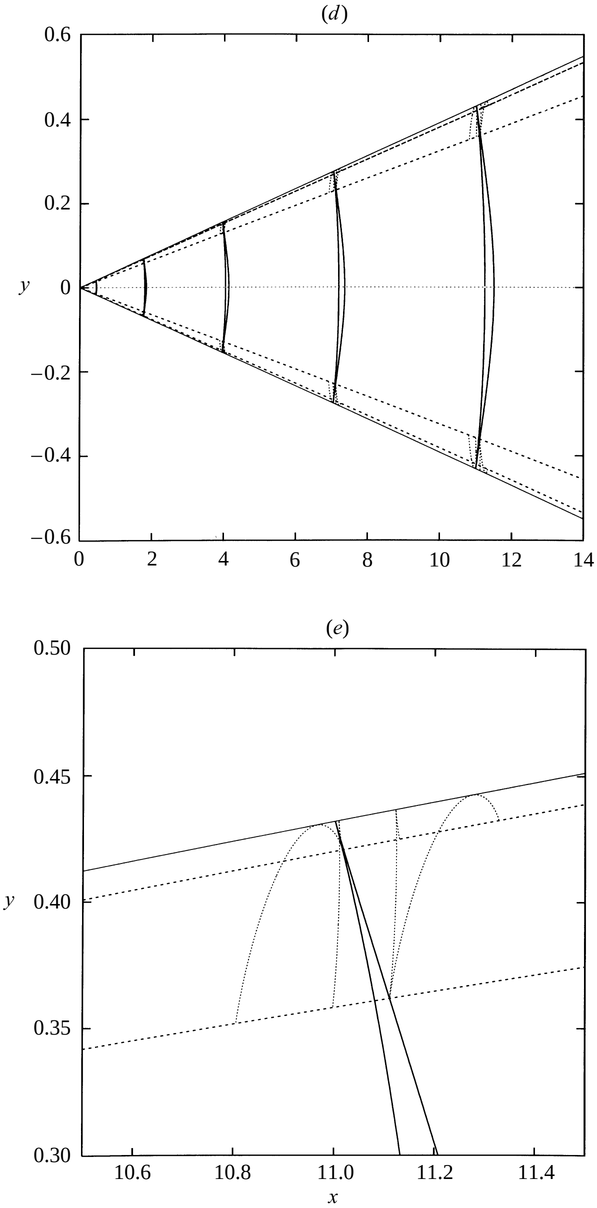


Figure 9. A few representative self-similar flows with $(A, B) = (0.7, 0.001)$ and $\mu = -2$: (*a*) fast flow with $\Omega = 1.3$; (*b*) fast flow with $\Omega = \Omega_{cr} = 1.673$; (*c*) fast flow with $\Omega = 1.9$; (*d*) slow flow with $\Omega = 2.5$; (*e*) a detail of (*d*) - - - - - -, transition; ——, limit; ▬▬, streamline; ······, characteristic. Only flow patterns in the right half-plane are shown.

8. Conclusions

In this paper we have proposed a new method for studying steady plasma flows, which is a natural extension of the classical (but seldom used) method developed for describing transonic fluid flows. We have considered plasma flows in intrinsic (natural) coordinates and obtained three scalar nonlinear equations of mixed elliptic–

hyperbolic type describing such flows. These equations are much more convenient to use than the conventional coupled system of two equilibrium equations (one differential and one algebraic) in physical coordinates. As might be expected, we have to pay a price for such a simplification when we describe equilibria found in the intrinsic coordinates in the physical coordinates. Dealing with the boundary conditions is particularly difficult is this formulation. We have mainly used our intrinsic equations to describe some *explicit* steady flows either in the entire symmetry plane or in its appropriate subdomains. To illustrate the possibilities of our method, we started with constructing the MHD counterparts of the celebrated Prandtl–Meyer fans and Ringleb flows around semi-infinite plates. Next, we addressed the question of finding general self-similar solutions from the group-theoretical viewpoint, and described the most general condition for such solutions to exist. Assuming that this condition is satisfied, we have derived an ODE governing self-similar solutions. We have presented a general discussion of the properties of this ODE, and described certain cases when it can be solved in a closed form. As a result, we have obtained a detailed description of representative (albeit special) classes of steady plasma flow. Not only are these flows interesting in their own right, but they can also (in all likelihood) be considered as intermediate asymptotics for time-dependent solutions of the MHD equations. Only stable (or weakly unstable) steady flows are of physical interest. While there are several powerful methods available for studying the stability of static equilibria both analytically and numerically, for steady plasma flows the situation is less satisfactory. The stability investigation for self-similar flows described in this paper is an interesting challenge.

Several important issues concerning transonic MHD flows remain to be addressed. First, a convenient way of formulating boundary conditions in intrinsic coordinates has to be found. Secondly, an adequate mathematical theory for the corresponding boundary-value problems has to be created, and novel numerical methods for solving them under concrete circumstances need to be developed, especially in order to study interactions of plasma flows with conducting bodies and plasma flows in channels. We expect that the novel equilibrium equation (42) is easier to deal with for these purposes than the original coupled equations (10) and (11). In addition, it is interesting to extend the method of this paper to flows with axial and helical symmetry.

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